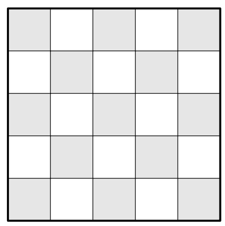
Parabola Volume 61 Issue 2 (2025), Article 11

Problems 1771–1780

Parabola would like to thank Kirill Bairak, Barnaul, Russia for contributing **Q1774**.

Q1771 A follow-up to **Q1764**. Consider sequences of positive integers $x_1, \ldots, x_{100}$ with the property that either $x_{k+1} = 4x_k$ or $x_k = 5x_{k+1}$ for each $k = 1, \ldots, 99$. What is the smallest possible sum $x_1 + \cdots + x_{100}$ for such a sequence?

Q1772 In the game of chess, a knight can move from its current square to any square reached by going two squares horizontally or vertically, then one square in a perpendicular direction. There are 13 black squares on a 5×5 chessboard,



and no two of them are separated by a knight's move; moreover, as we showed in the past problem **Q1705** (*Parabola* Volume 59 Issue 1 (2023)), there is no other choice of 13 squares for which this is true.

Now suppose that we want to choose 12 squares on the 5×5 board, such that no two are separated by a knight's move. Clearly we could take all 12 white squares, or any 12 of the 13 black squares. Is there any other way to do this? That is: is it possible to choose 12 squares on a 5×5 chessboard, not all of the same colour, such that no two are separated by a knight's move?

Q1773 We have an unlimited supply of cards which are red on one side, an unlimited supply of green, and one yellow. We create a row of n cards consisting, from left to right, of some number (possibly 0) of green, one yellow and some number (possibly 0) of red. We turn all the cards over so that the colours are unseen, then invite Heidi into the room. She knows all the above information (but does not know how many red and green cards there are). She is allowed to turn over up to 10 cards, one at a time, and she wins if she turns over the yellow card; except that she loses if she turns over four red cards. What is the maximum value of n for which Heidi can be *certain* of winning this game?

Now suppose that there is a row of 100 cards and that Heidi can still, as above, turn over three reds without losing. How many turns does Heidi need to win the game, and how does she do it?

Q1774 Let $\triangle ABC$ be a triangle in which $|AB| \neq |AC|$, let I be its in-centre, and let E be the foot of the perpendicular from I to side BC. The line through A and I intersects the circumcircle of $\triangle ABC$ again at a point K. The line KE intersects the circumcircle again at point F. Prove that the following three circles have a common point: the circumcircle of $\triangle ABC$, the circle with KI as its diameter, and the circumcircle of $\triangle FIE$.

Q1775 Show that for any positive integer n, and for any $2n$ real numbers

$$a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$$

lying between -1 and 1 (inclusive), we have

$$|a_1 a_2 \cdots a_n - b_1 b_2 \cdots b_n| \leq |a_1 - b_1| + |a_2 - b_2| + \cdots + |a_n - b_n| \,.$$

Q1776 A virus is located at a point of an infinite two–dimensional grid of 1×1 squares. One can heal a virus at a point if it has at least four non–infected adjacent points – "adjacent" means adjacent either horizontally, vertically or diagonally. However, healing a virus at one point leads to infecting four adjacent points. Is it possible to reach a situation in which there are no viruses at distance 3 or less from the location of the original virus?

Q1777 Let n be an integer, $n \geq 5$. Prove that if the distinct prime factors of n are $p_1, p_2, \ldots, p_k$, then the product

$$A = n^2\Big(1 - \frac{1}{p_1^2}\Big)\Big(1 - \frac{1}{p_2^2}\Big) \cdots \Big(1 - \frac{1}{p_k^2}\Big)$$

is equal to 24 times an integer.

Q1778 Jack is playing cards with three other players seated around a table. A pack of 52 cards is being used. Jack deals cards to the person on his left, then to the next person around the table, and so on. But instead of dealing single cards, he gives 5 to the first person, 2 to the next, then $1, 1, 1, 2, 1$; and then, continuing around the table, repeats the sequence $5, 2, 1, 1, 1, 2, 1$ as many times as needed until the whole pack has been dealt out. Jack is amazed to find that despite this irregular deal, each of the four players ends up with the same number of cards: a fact that you can easily confirm.

Next day, Jack is playing again and decides on another unconventional way of dealing. He has in mind a sequence of non–negative integers $a_1, a_2, \ldots, a_n$ which add up to 13. He deals cards in batches of $a_1, a_2, \ldots, a_n$ to successive players; and then continues around the table with the same sequence repeatedly until all 52 cards have been dealt. Prove that if n is odd, then every player ends up with the same number of cards; while if n is even, then this is not so.

Q1779 Simplify the sum

$$S_n = 1^2\binom{n+1}{1}^2 + 2^2\binom{n+1}{2}^2 + \cdots + (n+1)^2\binom{n+1}{n+1}^2 .$$

Q1780 Consider the cubic polynomial

$$f(x) = x^3 + \alpha x^2 + (\alpha - \beta)x - 1 \,,$$

where α and β are real numbers.

(a) Let $\beta = 2$. Find a value of α for which $f(x)$ has one rational root and two irrational roots.

(b) Find a value of β with the following property: for every real α, either $f(x)$ has three rational roots, or $f(x)$ has three real irrational roots.