Parabola Volume 61 Issue 2 (2025), Article 12

Solutions 1761–1770

Q1761 Let a be an integer. Find the number of integers b such that the quadratic

$$(x+a)(x+b)+2025$$

can be factorised as the product of two linear factors with integer coefficients.

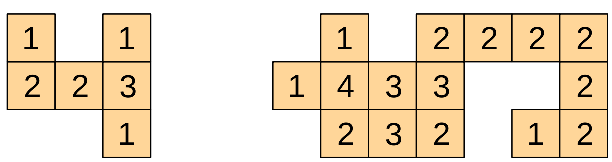
Comment. Isn't this basically the same as a question we asked last year? Don't be hasty…

SOLUTION By writing

$$(x+a)(x+b)+2025=(x+a+r)(x+a+s)$$

and following the procedure of **Q1740**, we find that the number of possibilities for (r,s) is equal to the number of factors of $2025=3^4\times 5^2$, which is 30. However, since 2025 is a square, interchanging r and s *does not* always give a different pair. Specifically, $r=45$, $s=45$ gives $s=45$, $r=45$ which is the same pair; and this is also true for $r=s=-45$. So each of these factorisations gives one value for b; the remaining 28 factorisations can be grouped into 14 pairs giving 14 values of b; and the total number of possibilities for b is 16.

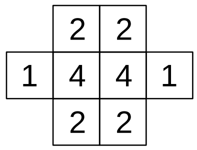
Q1762 A *polyomino* is a connected arrangement of squares, all the same size, and never meeting horizontally or vertically except along the full length of a side. Here are two examples; in each example, every square is labelled with the number of squares which adjoin it.

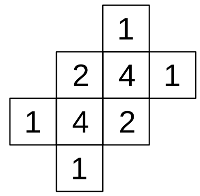


Is there an arrangement of this type in which the numbers $1,2,3,4$ all occur equally often? If so, then find a smallest possible example.

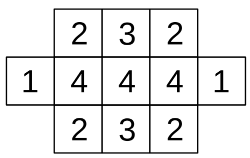
Thanks to user Evan at math.stackexchange.com for inventing this problem and allowing *Parabola* to publish it.

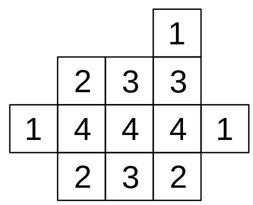
SOLUTION It is clear that the number of squares in such an arrangement must be a multiple of 4. It is impossible to have just 4 squares as none of them could be adjacent to 4 others. If there are 8 squares, then exactly two of them must be labelled 4. Firstly, suppose that these two are contiguous. This fully determines the rest of the arrangement, see below left, and it doesn't work. If the two "4–squares" are not contiguous, then these squares and their neighbours amount collectively to 10 squares; to reduce this to 8 there must be 2 overlap squares; the only possibility is shown below right, and again it doesn't work.





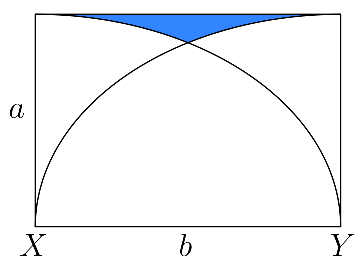
We investigate the possibility of a 12–square polyomino. If we put the three "4–squares" together in a line we obtain the arrangement of eleven squares below left. We have to add one more square in such a way as to convert one of the "2–squares" to a "3"; and the extra square should itself be a "1". This is not hard by trial and (no) error, and our result is shown below right.



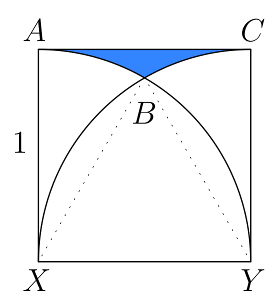


There are other possible 12–square polyominos which also work: we invite readers to explore further.

Q1763 The diagram shows an a by b rectangle containing two quarters of ellipses with centres at X and Y. Calculate the area of the blue shaded region.



SOLUTION If the rectangle is replaced by a 1×1 square, then the ellipses become circles of radius 1;



triangle XBY is equilateral; regions XAB and YCB are 30° sectors of the circles; and the shaded area is the area of the square, minus the areas of the triangle and the two sectors,

$$1 - \frac{\sqrt{3}}{4} - \frac{\pi}{6}\ .$$

If we now change the square back to an $a \times b$ rectangle, then all areas will be multiplied by a in one direction and by b in a perpendicular direction. So the area of the blue region in the problem is

$$ab\left(1 - \frac{\sqrt{3}}{4} - \frac{\pi}{6}\right)\ .$$

Q1764 We wish to find sequences of positive numbers $x_1, x_2, \ldots, x_{18}$ such that

- for every k from 1 to 17, we have either $x_{k+1} = 4x_k$ or $x_k = 5x_{k+1}$;
- $x_1 + x_2 + \cdots + x_{18} = 2025$.

How many such sequences are there? Show that **none** of the numbers $x_1, x_2, \ldots, x_{18}$ in such a sequence can be an integer.

SOLUTION For $k = 1, 2, \ldots, 17$, let r_k be the ratio x_{k+1}/x_k. Note that $x_2 = r_1x_1$, and $x_3 = r_2x_2 = r_2r_1x_1$, and in general

$$x_k = r_{k-1}r_{k-2}\cdots r_1x_1\ . \tag{$*$}$$

Now every r_k has one of two values, 4 or $\frac{1}{5}$, and there are therefore 2^{17} choices for the sequence $r_1, r_2, \ldots, r_{17}$. For each such sequence, we have

$$(1 + r_1 + r_2r_1 + \cdots + r_{17}r_{16}\cdots r_1)x_1 = x_1 + x_2 + x_3 + \cdots + x_{18} = 2025\ ,$$

which determines the value of x_1; and then $(*)$ determines the values of all other x_k. So there are 2^{17} possible choices for the sequences $x_1, x_2, \ldots, x_{18}$, and this answers the first part of the question.

To investigate the possibility of integer values for x_k, we would prefer to avoid equations involving fractions. So we multiply 5^{17} into the previous equation, obtaining

$$\big(5^{17} + 5^{16}(5r_1) + 5^{15}(5r_2)(5r_1) + \cdots + (5r_{17})(5r_{16})\cdots(5r_1)\big)x_1 = 2025 \times 5^{17}\ ;$$

and then multiply by $r_{k-1}r_{k-2}\cdots r_1$, giving

$$\begin{aligned}\big(5^{17} + 5^{16}(5r_1) + 5^{15}(5r_2)(5r_1) + \cdots + (5r_{17})(5r_{16})\cdots(5r_1)\big)x_k& \\ = 2025 \times 5^{17-(k-1)}(5r_{k-1})(5r_{k-2})\cdots(5r_1)\ .&\end{aligned}$$

Now suppose that x_k is an integer. All the bracketed terms $(5r_j)$ are integers; in fact, each of them is either 1 or 20; and so we can read this equation modulo 19, which gives

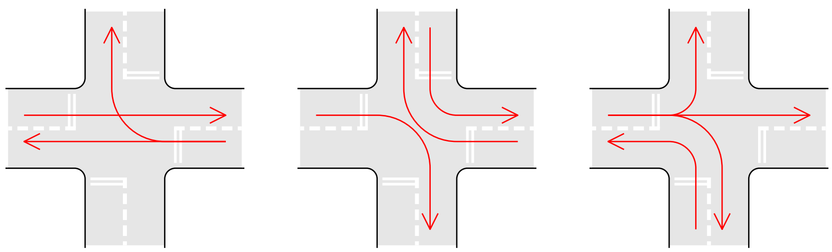
$$\big(5^{17} + 5^{16} + 5^{15} + \cdots + 1\big)x_k \equiv 2025 \times 5^{18-k} \pmod{19}\ .$$

Now sum the geometric series on the left–hand side (and multiply by 4 for convenience):

$$(5^{18}-1)x_k \equiv 4\times 2025\times 5^{18-k} \pmod{19}\ .$$

But a very important result of number theory known as *Fermat's Little Theorem* (look it up!) shows that $5^{18}-1$ is a multiple of 19; therefore $4\times 2025\times 5^{18-k}$ is a multiple of 19. But it is obvious that neither 4 nor 5 is a multiple of 19; and it is a simple calculation to check that 2025 is not a multiple of 19 either; so this is impossible. Hence, x_k cannot be an integer, and we are finished.

Q1765 A group of traffic engineers is designing a traffic–light sequence for an intersection of two two–way roads. Any vehicle approaching the intersection will see lights showing whether travel straight on, or left, or right is permitted. (No U–turns!) An allowable traffic flow must be such that there is no possibility of collisions between cars travelling through the intersection at the same time, as long as they are obeying the lights. Three traffic flows are shown below.



The first of these is not allowable because a car turning from the eastern road to the northern could collide with one travelling straight from the western road to the eastern. The second and third traffic flows are OK.

How many allowable traffic flows are there? (**Note**. This puzzle is published in Australia, so cars must drive on the left side of the road!)

SOLUTION There are four possible lanes on which cars may leave the intersection. No two cars can use the same departure lane, as this could cause a collision, so any allowable pattern consists of at most one route into each departure lane. Moreover, if we have an allowable pattern consisting of straight–ons and right–turns, then we can (if we wish) add left turns into any unused lanes, since these are in no danger of colliding with a vehicle entering any other lane. Therefore, we can answer the question by determining how many allowable traffic flows there are which involve only straight–on paths and right turns, and then adding any selection of the available left turns. Note that

- three straight paths cannot occur together in the same traffic flow, and two can occur together only if they are going in opposite directions (example in the first diagram from the question);
- two right turns cannot appear together unless they come from opposite roads (second diagram in the question);

- a straight path and a right turn cannot appear in the same traffic flow unless they start in the same lane (third diagram).

It is now not too hard to tabulate all possibilities. For example, if we have one straight path and no right turns, then there are 4 options for the straight path; there are three empty departure lanes, and we can choose any selection of these (2^3 possibilities) to receive left turns. Similar considerations give the following.

straight paths	right turns	number of flows
0	none	2^4
	one	4×2^3
	two, must be opposite	2×2^2
1	none	4×2^3
	one	4×2^2
2	none	2×2^2

Altogether, there are 112 possible traffic flows.

Q1766 Holly and Ted take turns throwing a coin, with Holly going first. Holly wins if she throws a head; Ted wins if he throws a tail. (Once one of these events occurs, the game is over and there are no further throws.) However, the coin is not known to be a fair coin: that is, the probability p of throwing a head might not be $\frac{1}{2}$. Find the value of p for which Ted has an even chance of winning this game.

SOLUTION For convenience we write $q = 1-p$ for the probability of throwing a tail. It is clear that if $p = 1$ then Holly wins for sure, and if $q = 1$ then Ted wins for sure; so we may ignore these possibilities. Now Ted wins the game in the following circumstances:

- Holly throws a tail, Ted throws a tail: probability q^2;
- tail, head, tail, tail: probability pq^3;
- tail, head, tail, head, tail, tail: probability p^2q^4;
- and so on.

Therefore, Ted's total winning probability is obtained by summing a geometric series,

$$q^2 + pq^3 + p^2q^4 + p^3q^5 + \cdots = \frac{q^2}{1-pq} = \frac{q^2}{1-q+q^2} \ ,$$

and he has an even chance of winning if

$$\frac{q^2}{1-q+q^2} = \frac{1}{2} \quad \Leftrightarrow \quad q^2 + q - 1 = 0 \quad \Leftrightarrow \quad q = \frac{-1+\sqrt{5}}{2} \ ,$$

where we have rejected the negative solution for q.

Check. This gives $q \approx 0.618$, which makes sense: Holly has an advantage by virtue of throwing first; for Ted to have an even chance this must be offset by having a tails probability greater than $\frac{1}{2}$.

Since the question asked for the value of p, the answer is

$$p = 1 - q = \frac{3 - \sqrt{5}}{2} \,.$$

Alternative solution. Suppose that each player has an even chance of winning. If Holly throws a head, then she wins straight away, probability p. If she throws a tail, then Ted must throw a head for Holly to have any chance of winning. The probability of this "branch" so far is $(1-p)p$. But Holly is now in exactly the same position she was at the beginning, so her winning probability from here on is $\frac{1}{2}$. Therefore, we have

$$p + (1-p)p\Big(\frac{1}{2}\Big) = \frac{1}{2} \,.$$

Simplifying, we get a quadratic for p; one root of the quadratic is greater than 1 and is therefore not an admissible probability; the other root is $p = \frac{1}{2}\big(3 - \sqrt{5}\big)$ as above.

Q1767 Let S be a set of numbers from 0 to 1 which has the following properties.

(1) The numbers 0 and 1 are in S.

(2) Whenever x and y are numbers in S, their average $(x+y)/2$ is also in S.

Find an example in which S **is not** the set of all numbers from 0 to 1. Then prove that if there is a number a with $0 < a < 1$, such that S contains all numbers from a to 1, then S **is** the set of all numbers from 0 to 1.

SOLUTION Let S be the set of all fractions from 0 to 1 in which the denominator is a power of 2: that is,

$$S = \left\{ \frac{m}{2^n} \;\middle|\; m, n \text{ are integers},\ n \ge 0,\ 0 \le m \le 2^n \right\} .$$

The average of any two numbers in this set is given by

$$x = \frac{m_1}{2^{n_1}} \,, \quad y = \frac{m_2}{2^{n_2}} \,, \quad \frac{x+y}{2} = \frac{m_1 2^{n_2} + m_2 2^{n_1}}{2^{n_1+n_2+1}} \;;$$

this average is a fraction whose denominator is a power of 2, and is therefore an element of S. So S has the stated properties. However, the fraction $\frac{1}{3}$ (for example) is not in S because its denominator is not a power of 2. So S is not the set of all numbers from 0 to 1.

Now let a be any number between 0 and 1, and assume that S contains all numbers from a to 1. We need to prove that S also contains all numbers x between 0 and a. Now if y is in S, then so are all of the following numbers:

$$\frac{0+y}{2} = \frac{y}{2} \;;$$

$$\frac{0+(y/2)}{2}=\frac{y}{4} \quad\text{and}\quad \frac{(y/2)+y}{2}=\frac{3y}{4}\ ;$$
$$\frac{0+(y/4)}{2}=\frac{y}{8} \quad\text{and}\quad \frac{(y/4)+(y/2)}{2}=\frac{3y}{8}\ ;$$
$$\text{and}\quad \frac{(y/2)+(3y/4)}{2}=\frac{5y}{8} \quad\text{and}\quad \frac{(3y/4)+y}{2}=\frac{7y}{8}\ ;$$

and it is not hard to show that S contains all numbers $my/2^n$ with $0 \le m \le 2^n$. So, to prove that any given number x between 0 and a is in S, it suffices to write it as

$$x=\frac{my}{2^n}\ ,$$

where m, n are integers with $n \ge 0$ and $0 \le m \le 2^n$, and $a < y < 1$. In other words, we want

$$a < \frac{2^n x}{m} < 1\ ,$$

or, rearranging,

$$2^n x < m < \frac{2^n x}{a}\ .$$

Such an integer m is guaranteed to exist if the upper and lower bounds here differ by more than 1. So, given x with $0 < x < a < 1$, we can do the following. Choose n such that

$$2^n > \frac{1}{x\left(\dfrac{1}{a}-1\right)}\ .$$

Then

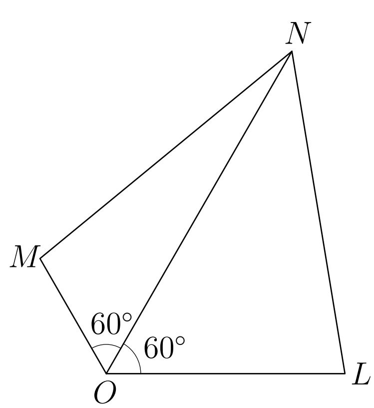
$$\frac{2^n x}{a} - 2^n x > 1\ ,$$

so there exists an integer m with

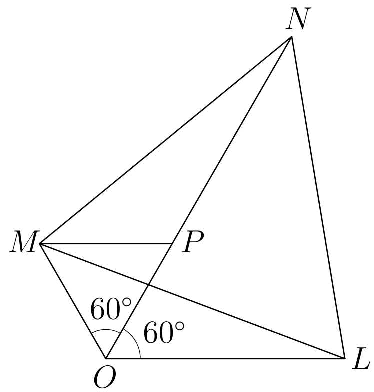
$$0 \le 2^n x < m < \frac{2^n x}{a} \le 2^n\ .$$

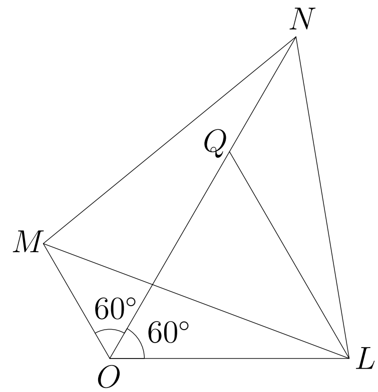
Let $y = 2^n x/m$; then we have $a < y < 1$, so by assumption, y is in S. From the above argument, we know that $my/2^n$ is in S; that is, x is in S, and we are finished.

Q1768 In the diagram there are angles of 60° as marked; also, $ON = OM + OL$. Prove that $\angle MNL = 60^\circ$.



SOLUTION We have copied the diagram twice and made some extra constructions.





In the first diagram, P is the point on ON such that $OP = OM$; this means that $PN = OL$ and $\triangle OMP$ is equilateral. In the second diagram, $OQ = OL$; so $QN = OM$ and $\triangle OLQ$ is equilateral. Therefore, we have

- $\angle NPM = 120° = \angle LOM$ and $PN = OL$ and $PM = OM$;
- so $\triangle NPM$ and $\triangle LOM$ are congruent;
- $\angle LQN = 120° = \angle LOM$ and $QL = OL$ and $QN = OM$;
- so $\triangle LQN$ and $\triangle LOM$ are congruent.

From the two congruencies we have $ML = MN$ and $ML = NL$; therefore, $\triangle LMN$ is equilateral, and $\angle MNL = 60°$ as required.

Alternative solution, by coordinate geometry. Let O be the origin and OL the x–axis; write $OL = x$ and $OM = y$. Then the points L, M, N have coordinates

$$\begin{aligned}
L &= (x, 0) \\
M &= \big(y\cos 120°,\ y\sin 120°\big) = \Big(-\frac{y}{2},\ \frac{y\sqrt{3}}{2}\Big) \\
N &= \big((x+y)\cos 60°,\ (x+y)\sin 60°\big) = \Big(\frac{x+y}{2},\ \frac{(x+y)\sqrt{3}}{2}\Big)\ ,
\end{aligned}$$

and we can calculate distances

$$\begin{aligned}
LM^2 &= \Big(x + \frac{y}{2}\Big)^2 + \Big(\frac{y\sqrt{3}}{2}\Big)^2 = x^2 + y^2 + xy \\
MN^2 &= \Big(\frac{x+2y}{2}\Big)^2 + \Big(\frac{x\sqrt{3}}{2}\Big)^2 = x^2 + y^2 + xy \\
NL^2 &= \Big(\frac{-x+y}{2}\Big)^2 + \Big(\frac{(x+y)\sqrt{3}}{2}\Big)^2 = x^2 + y^2 + xy\ ,
\end{aligned}$$

showing again that $\triangle LMN$ is equilateral. Hence, $\angle MNL = 60°$.

Q1769

(a) Prove that if x is any real number, then

$$\sin 2x \sin 9x + \sin 3x \sin 4x - \sin 5x \sin 6x = 0 \, .$$

(b) Prove that there are no positive numbers a, b, c, d, e, f such that

$$\sin ax \sin bx + \sin cx \sin dx + \sin ex \sin fx = 0$$

for all real values of x.

SOLUTION If we use the products–to–sums formula

$$2 \sin A \sin B = \cos(A - B) - \cos(A + B) \, ,$$

then twice the left–hand side in (a) is

$$(\cos 7x - \cos 11x) + (\cos x - \cos 7x) - (\cos x - \cos 11x) = 0 \, .$$

For (b), choose x to be a positive number less than all of the (positive) quantities

$$\frac{\pi}{a} \, , \; \frac{\pi}{b} \, , \; \frac{\pi}{c} \, , \; \frac{\pi}{d} \, , \; \frac{\pi}{e} \, , \; \frac{\pi}{f} \, .$$

Then ax, bx and so on are positive numbers less than π; so $\sin ax, \sin bx$ and so on are positive; and the given sum must be greater than zero.

Alternatively, a calculus solution. Denote the left–hand side of the equation in (b) by $f(x)$. If $f(x) = 0$ for all x, then the second derivative $f''(x)$ is also zero for all x; specifically, $f''(0) = 0$. However,

$$\frac{d}{dx}\bigl(\sin ax \sin bx\bigr) = a \cos ax \sin bx + b \cos bx \sin ax \, ,$$

so

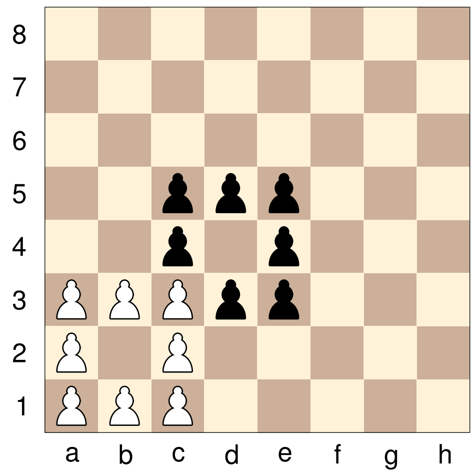
$$\frac{d^2}{dx^2}\bigl(\sin ax \sin bx\bigr) = -a^2 \sin ax \sin bx + 2ab \cos ax \cos bx - b^2 \sin bx \sin ax \, ,$$

and the latter equals $2ab$ when $x = 0$. Treating the other terms in $f(x)$ similarly, we find

$$f''(0) = 2ab + 2cd + 2ef \, ,$$

which cannot be zero because a, b, c, d, e, f are all positive. Therefore, $f(x)$ cannot be zero for all x.

Q1770 Consider the following placement of eight white pawns and seven black pawns on a chessboard. (**Note** that this is not going to be a chess problem, so it does not matter that the position is illegal!)



In the game of chess, a knight can move from its current square to any square reached by going two squares horizontally or vertically, then one square in a perpendicular direction. It captures a pawn by moving onto the pawn's square. In this problem, a grey knight (it is grey so that it can capture both white and black pawns) was placed on the chessboard, on a square not occupied by any pawn; then made 15 consecutive moves capturing all 15 pawns; and then one more move returning to its initial square. The first and last pawns captured by the knight were of the same colour. On which square did the knight start?

SOLUTION If we temporarily ignore the knight's first and last moves, we need to find a sequence of 14 knight's moves which begins on one of the occupied squares and visits all the others once each. Observe that a2 is a knight's move away from just two of the occupied squares, and one of these two squares is c3; and that exactly the same is true for b1, d5 and e4. Only two of these four squares can come immediately before or after c3; so the other two can only access one other occupied square and therefore must be the locations of the first and last pawns captured. Since these two pawns were of the same colour, there are just two options: a2 and b1, or d5 and e4.

The starting square of the knight, which is the same as its final square, must be a knight's move from both the first and last pawns. If these are d5 and e4, the possible starting squares are c3 and f6; but the former must be ruled out as it is initially occupied by a pawn. If the first and last pawns to be captured are a2 and b1, then c3 must be ruled out for the same reason; and the only other "potential" square would be off the edge of the board, so it is not admissible. Therefore, the only possible starting square for the knight is f6.

To confirm that the task we have described can actually be achieved, here is one possible sequence of moves:

f6–d5–e3–c4–e5–d3–c1–a2–c3–b1–a3–c2–a1–b3–c5–e4–f6 .