Parabola Volume 61, Issue 2 (2025)

A trailer for the Extension III Mathematics course

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1 About the Extension III Mathematics course

The Extension III mathematics course is aimed at talented high school students who have finished the high school curriculum early and want to see more interesting mathematics before going to university. The materials can be found at

`https://web.maths.unsw.edu.au/~danielch/ExtensionIII.html`

The course is designed to be self-studied. To complement both the high school and university mathematics curricula, I have chosen the topic of recurrence relations and generating functions. If you don't know much about these topics, then read on for a little trailer to the course!

2 Recurrence relations

Sequences in mathematics are often constrained by equations called *recurrence relations*. Rather than defining what this is, we will give a simple example. The following comes from the classic "gambler's ruin" problem.

Suppose Donald needs \$ W but only has \$ n. He decides to play a betting game at a casino. The game involves repeatedly tossing a coin and betting a dollar on the outcome. He wins a dollar whenever it comes up heads but loses the dollar he bet when it comes up tails. The game ends when either he finally loses all his cash, or he wins and walks away with \$ W. We wish to determine the probability p_n that he wins.

The secret to answering this question is to fix W and consider the sequence of probabilities $p_0, p_1, \ldots, p_W$. Note that $p_0 = 0$ and $p_W = 1$. Suppose that $0 < n < W$. Then after one toss of the coin we find that half the time, it comes out heads in which case the probability he wins thereafter is p_{n+1}. The other half of the time, he wins with probability p_{n-1}. Thus, $p_n = \frac{1}{2}(p_{n+1} + p_{n-1})$. We can re-write this as

$$p_{n+1} = 2p_n - p_{n-1}\,. \tag{1}$$

This is an example of a *recurrence relation* since it expresses the terms of the sequence in terms of the previous two. In particular, given the values p_0, p_1, one can use the recurrence relation to determine all the other values of the sequence recursively. In fact, it will also be useful to define p_n for $n > W$ by assuming Equation (1) holds for $n \geq W$ as well.

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3 Generating functions

It turns out that this recurrence relation is rather special so can be solved by elementary means (can you find the solution?). We will however introduce a general technique for studying recurrence relations, that of *generating functions*. In this case, this is the function, defined by the *power series*

$$P(x) = \sum_{n=0}^{\infty} p_n x^n \,. \tag{2}$$

If we can determine $P(x)$, then we presumably can find the sequence by extracting the coefficients of the series. Consider

$$\begin{aligned}
(1 - 2x + x^2)P(x) &= \sum_{n=0}^{\infty} p_n x^n - 2\sum_{n=0}^{\infty} p_n x^{n+1} + \sum_{n=0}^{\infty} p_n x^{n+2} && (3)\\
&= p_0 + p_1 x + \sum_{n=2}^{\infty} p_n x^n - 2(p_0 x + \sum_{n=2}^{\infty} p_{n-1} x^n) + \sum_{n=2}^{\infty} p_{n-2} x^n && (4)\\
&= p_1 x + \sum_{n=2}^{\infty} (p_n - 2p_{n-1} + p_{n-2}) x^n && (5)\\
&= p_1 x && (6)
\end{aligned}$$

where we used $p_0 = 0$ to obtain Equation (5) and the recurrence relation (1) to obtain Equation (6).

It is now easy to find

$$P(x) = \frac{p_1 x}{(1-x)^2} = p_1 x \frac{d}{dx}\frac{1}{1-x} = p_1 x \frac{d}{dx}(1 + x + x^2 + \cdots) = p_1 x(1 + 2x + 3x^2 + \cdots)$$

where we assume that we can differentiate the geometric series using the usual differentiation of a polynomial formula. Now, computing the coefficient of x^n gives $p_n = p_1 n$. Since $1 = p_W = p_1 W$, we also know that $p_1 = \frac{1}{W}$, so the final answer is $p_n = \frac{n}{W}$.

Elegant though this computation is, you may wonder how to make the steps legitimate. Check out the course to find out about this as well as many other applications and extensions of this method!