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Numerically computed double and triple bubbles in $\mathbb{R}^3$ for density r^p

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1 Introduction

The isoperimetric problem is one of the oldest in mathematics. It asks for the least-perimeter way to enclose given volume. For a single volume in Euclidean space (with uniform density) of any dimension, the well-known solution is any sphere. With density r^p, Boyer et al. [5] found that the solution for a single volume is a sphere through the origin. For two volumes in Euclidean space, Hutchings et al. [1] showed that the standard double bubble, consisting of three spherical caps meeting along a curve in threes at $120°$ angles as in Figure 1, provides the isoperimetric cluster. Recently, Milman and Neeman [10] showed that the standard triple bubble is the isoperimetric cluster (see Figure 1).

In this paper we use Brakke's Evolver [2] to numerically compute double and triple bubbles in $\mathbb{R}^3$ with density r^p for various $p > 0$, extending many of the results of Collins [3] from $\mathbb{R}^2$ to $\mathbb{R}^3$. Some videos are downloadable from Google Drive.[2] Proposition 7 supports the conjecture of Hirsch et al. [8] that the optimal double bubble is the standard double bubble with the singular circle passing through the origin (see Figure 3). Proposition 8 indicates that the optimal triple bubble resembles a standard triple bubble (Figure 1) with one vertex at the origin (see Figure 4).

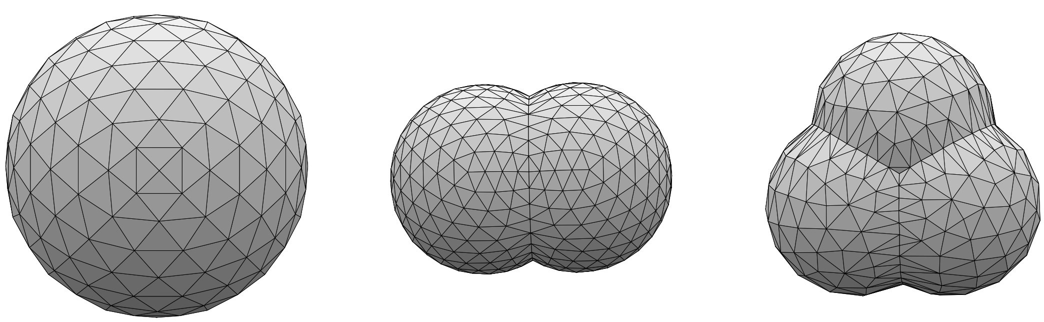

Figure 1: The optimal single, double [1] and triple bubbles [10] with equal volumes in Euclidean space.

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[2]https://drive.google.com/drive/folders/1-3AJLlG08R-lC1MB2gCfqxSAUm48EPOx?usp=sharing

History

Examination of isoperimetric regions in the plane with density r^p began in 2008 when Carroll et al. [4] showed that the isoperimetric solution for a single area in the plane with density r^p is a convex set containing the origin. It was something of a surprise when Dahlberg et al. [6] proved that the solution is a circle through the origin. In 2016, Boyer et al. [5] extended this result to higher dimensions. In 2019, Huang et al. [7] studied the 1-dimensional case, showing that the best single bubble is an interval with one endpoint at the origin and that the best double bubble is a pair of adjacent intervals which meet at the origin. Ross [9] showed that, in $\mathbb{R}^1$, multiple bubbles start with the two smallest meeting at the origin and the rest in increasing order alternating side to side.

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2 Definitions

Definition 1 (Density Function)**.** *Given a smooth Riemannian manifold M, a* density *on M is a positive continuous function (perhaps vanishing at isolated points) that weights each point $p \in M$ with a certain mass $f(p)$. Given a region $\Omega \subset M$ with piecewise smooth boundary, the weighted volume (or area) and boundary measure or perimeter of Ω are given by*

$$V(\Omega) = \int_\Omega f\, dV_0, \qquad P(\Omega) = \int_{\partial\Omega} f\, dP_0\,,$$

where dV_0 and dP_0 denote Euclidean volume and perimeter. We may also refer to the perimeter of Ω as the perimeter of its boundary.

Definition 2 (Isoperimetric Region)**.** *A region $\Omega \subset M$ is* isoperimetric *if it has the smallest weighted perimeter of all regions with the same weighted volume. The boundary of an isoperimetric region is also called isoperimetric.*

We generalise isoperimetric regions by considering two or more volumes.

Definition 3 (Isoperimetric Cluster)**.** *An isoperimetric cluster is a set of disjoint open regions Ω_i of volume V_i such that the perimeter of the union of the boundaries is minimised.*

For example, in $\mathbb{R}^3$ with density 1, optimal clusters are known for one volume, two volumes [1] and three volumes [10], as in Figure 1. Note that for density r^p, scalings of minimisers are minimisers, because scaling up by a factor of λ scales perimeter by λ^{p+2} and volume by λ^{p+3}.

The following proposition summarises existence, boundedness and regularity of isoperimetric clusters in $\mathbb{R}^n$ with density, proved by Hirsch et al. [8] following such results for single bubbles (Rosales et al. [11, Thm. 2.5] and Morgan [12]).

Proposition 4 (Existence, Boundedness and Regularity)**.** *Consider $\mathbb{R}^n$ with radial non-decreasing density f, smooth and positive except possibly at the origin, such that $f(r) \to \infty$ as $r \to \infty$. An isoperimetric cluster that encloses and separates given volumes exists ([8, Thm. 2.5]) and is bounded ([8, Prop. 2.6]). In $\mathbb{R}^3$, the cluster consists of smooth constant-generalised-mean-curvature surfaces meeting in threes at $120°$ along curves, which in turn meet in fours at $\cos^{-1}(-\frac{1}{3}) \approx 109°$ ([12, Sec. 13.9]), except possibly at the origin.*

Definition 5 (Generalized Curvature)**.** *We define, as in [11] and [**?**, Chap. 3], the (generalised) mean curvature of a surface with respect to the inward-pointing unit normal N as*

$$H_\psi = nH - \langle \nabla\psi, N \rangle\,,$$

where H is the classical mean curvature (the sum of principal curvatures) and ψ is the logarithm of the density function f. This comes from the first variation of perimeter formula, so that generalised curvature has the interpretation as minus the rate at which $\log f$ changes in the direction perpendicular to the curve (see [11, Sec. 3]).

Remark 6. In $\mathbb{R}^3$ with density r^p, a circular arc has constant generalised curvature if and only if the sphere passes through or is centred at the origin; see [8, Rem. 2.9].

3 Double and triple bubbles in $\mathbb{R}^3$ with density r^p

With Brakke's Evolver [2], Proposition 7 supports the conjecture of Hirsch et al. [8] that the optimal double bubble in $\mathbb{R}^3$ with density r^p is a standard double bubble with the singular circle passing through the origin. Proposition 8 provides a conjecture on the form of the triple bubble.

Proposition 7 (Double Bubble)**.** *Computations on Brakke's Evolver [2] support the conjecture [8] that the optimal double bubble in $\mathbb{R}^3$ for density r^p consists of a standard double bubble with the singular circle passing through the origin. See Figure 3.*

Proposition 8 (Triple Bubble)**.** *Computations with Brakke's Evolver [2] indicate that the optimal triple bubble in $\mathbb{R}^3$ with density r^p resembles a standard triple bubble with one vertex at the origin. See Figure 4.*

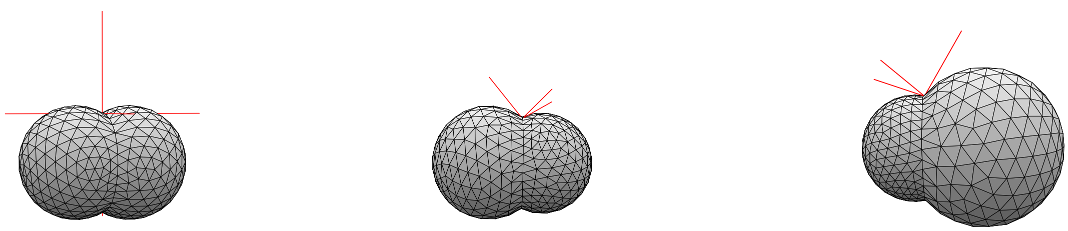
Remark 9. *The phenomenon of vertices of triple bubbles in $\mathbb{R}^2$ with density r^p approaching as p increases (Collins [3, Prop. 3.2]) apparently does not hold in $\mathbb{R}^3$.*

Proposition 10. *Our conjectured triple bubble of Proposition 8 has a lower surface area than three bubbles in a linear chain, as in Figure 2.*

(a) Our conjectured triple bubble with volumes 10, 10 and 1 has surface area just over 32.

(b) A linear chain with volumes 10, 10 and 1 has surface area just over 36.

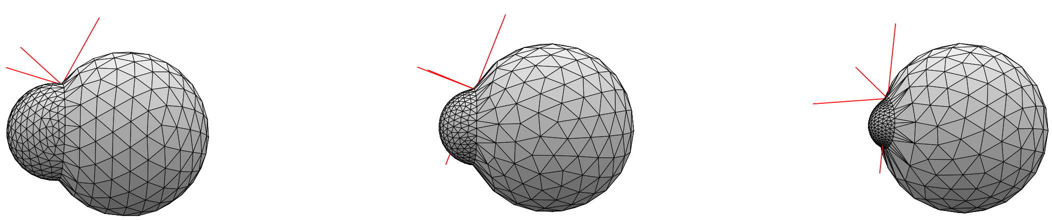
Figure 2: Our conjectured triple bubble has less surface area than a linear chain in $\mathbb{R}^3$ with density r^2. Densities $r^{0.5}$ and r^3 are apparently similar.



(a) Equal volumes

(b) Volumes 8 and 4

(c) Volumes 0.8 and 8

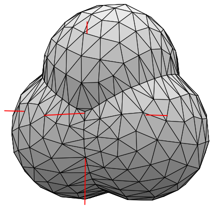


(d) Volumes 0.4 and 8

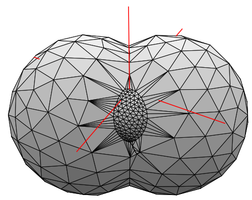
(e) Volumes 0.08 and 8

(f) Volumes 0.008 and 8

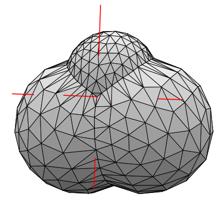
Figure 3: Computations in Brakke's Evolver [2] in $\mathbb{R}^3$ with density r^2 support the conjecture that the optimal double bubble is a standard double bubble with the singular circle passing through the origin (where the red axes meet). Densities r^5, r^3 and $r^{0.5}$ are apparently identical.



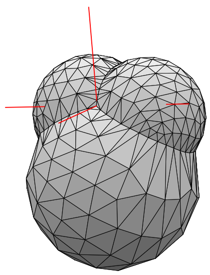
(a) Equal volumes of 300



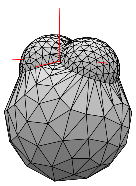
(b) Volumes 0.3, 300 and 300



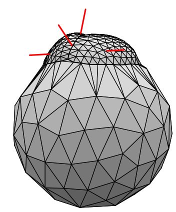
(c) Volumes 300, 300 and 30



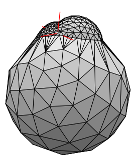
(d) Volumes 0.3, 0.3 and 3



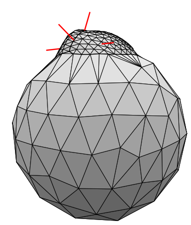
(e) Volumes 0.3, 0.3 and 15



(f) Volumes 0.3, 0.3 and 60



(g) Volumes 1, 5 and 1000



(h) Volumes 0.3, 0.3 and 300

Figure 4: Computations in Brakke's Evolver [2] in $\mathbb{R}^3$ with density r^2 suggest that the optimal triple bubble resembles a standard triple bubble with one vertex at the origin.

Figure 5: Computations in Brakke's Evolver [2] in $\mathbb{R}^3$ with densities $r^{0.5}$ and r^3 suggest that the optimal triple bubble resembles a standard triple bubble with one vertex at the origin, as for density r^2 of Figure 3.

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