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Exploding dice combinatorics

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1 Introduction

"Exploding dice" is a mechanic used in many popular tabletop games. The idea is simple. Whenever dice are rolled, some specific outcome on any of the dice (usually the highest possible value) allows you to continue rolling those dice and adding to your total, thereby making arbitrarily large sums possible. Several games have utilized this idea, but one game in particular has combined it with an interesting "push your luck" element.

Oathsworn: Into the Deepwood is a cooperative game designed by Jamie Jolly and published by Shadowborne Games in which 1-4 players journey together through a fascinating storyline punctuated by dice- and card-driven combat with various monsters. For more information about the game, see [5]. The first edition of the game was crowdfunded on Kickstarter.com in October of 2019 and received over $1.9 million in funding. A second edition received over $3.2 million in funding on Kickstarter in October of 2022. At the time of this writing, the game holds a rating of 8.9 on boardgamegeek.com and ranks #60 among all boardgames catalogued on the website.

At various points in the story, players are instructed to "perform a check". This means players must attempt to meet or exceed a given target sum by rolling any number of "*Oathsworn dice*" (see Figure 1). Each die has six faces. Two of the faces are blank (we'll say they're marked 0), two are marked with a 1, one is marked with a 2, and the last is marked with an outlined 2 (we'll refer to this as ②).

The rules are as follows:

(a) You may roll as many dice as you want, but if you roll two or more 0s, then the entire roll fails and the resulting sum is automatically considered zero.

(b) The result of your roll is simply the sum of all resulting numbers, with one exception. Whenever you roll a ②, not only do you add 2 points to the sum, you also get to roll that die again (and again and again and again if you keep rolling ②s). We'll call these *bonus rolls*. Note that rolling zeros on bonus rolls cannot cause your entire roll to fail. Only rolling two or more zeros on your initial roll results in a failure.

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Figure 1. Oathsworn Dice

For example, let's say the players are told that they each have to try to get a sum of at least six. The first player decides to roll five dice. When they do, they get the following:

This outcome can be denoted by 020②2. As you can see, the player got two 0s on their initial roll, so this attempt failed.

Now suppose the next player decides to roll four dice. They roll the following:

They now have a sum of 4 (with only one zero), and they get to roll that ② again.

Another ②! They have now reached the sum they need, so they have succeeded and can stop. However, if the target sum was higher, then they could roll that last one again. Say they did and they got a 1. Then they would stop and their outcome can be denoted by 1②01|②|1.

Now suppose the third player also decides to roll four dice. They roll the following:

They have a sum of 5, but they have two bonus rolls! They roll them and get

Since that the last zero happened on a bonus roll, it doesn't cause their entire roll to fail. Their sum is now 7, so they have succeeded. This outcome can be denoted by 10②②|02.

As you can see, the game presents an interesting dilemma. The more dice you roll, the better your chances of getting a higher sum. On the other hand, rolling more dice also means you increase the likelihood of getting at least two 0s on your initial roll, thereby causing the entire roll to fail. For some basic information on the mathematics of exploding dice, and an overview of the role that probability plays in the game *Oathsworn*, see Chapter 5 of [3].

For given positive integers k and n, the number of ways of getting a sum of k when rolling n Oathsworn dice can be quite large, and there isn't an obvious formula for computing it. In this article, we will seek an answer to the following question: in how many ways can one roll n Oathsworn dice and get a sum of k? We will use some very basic techniques of enumerative combinatorics (the mathematics of counting) to find an answer. These techniques are very common, and can be learned in introductory textbooks on combinatorics such as [1] or [2]. Perhaps one or more of the sequences we find will turn out to be known sequences!

2 The simple problems

One Oathsworn die

We're going to begin the way mathematicians often begin when trying to solve a new problem. We'll start with simple cases and develop some intuition that can then be applied to the more difficult cases. The simplest case of this problem is to imagine we're rolling just one Oathsworn die.

Let k and n be positive whole numbers. Define $S(n,k)$ to be the number of ways of rolling n Oathsworn dice and getting a sum of k. Clearly, $S(1,1) = 1$ and $S(1,2) = 2$ (the two possibilities are 2 and ②|0). Therein lies the only observation needed, since every odd result greater than 2 must end in a bonus roll of 1 and every even result greater than 2 must end in either a bonus roll of 2 or ②|0. Hence,

$$S(1,k) = \begin{cases} 1 & \text{if } k \text{ is odd;} \\ 2 & \text{if } k \text{ is even.} \end{cases}$$

That wasn't so bad. How difficult could this be?

Two Oathsworn dice

We now move onto the case in which two Oathsworn dice are rolled. From now on, we will assume that the dice we roll are all *distinguishable*. This is common when solving counting problems, as it can make counting easier. Specifically, it will allow us to imagine rolling the dice one at a time instead of all at once.

It is easy to find $S(2,1) = 2$ (01 and 10) and $S(2,2) = 5$ $(11, 20, 02,$ ②0|2 and 0②|0). For $k > 2$, further values of $S(2,k)$ can be found using a *recurrence relation*. A recurrence relation is a type of formula that uses earlier terms of a sequence to determine the next term. They can be convenient to set up because all you need is to think of a clever way of counting whatever it is you're trying to count. Imagine that you roll two dice one-at-a-time. There are four possibilities for that first roll (②, 2, 1 or 0). Depending on what happens, we can reason what needs to happen on the second roll and easily count the number of ways it can happen.

Imagine the first roll is a ②. Now you still get to roll both dice and must get a sum of $k-2$ in all possible ways. This is counted by $S(2,k-2)$. Now imagine the first roll is a 2. Then you have one die left to roll, and you have to get a sum of $k-2$ with that one die. The number of ways of doing that is counted by $S(1,k-2)$. Similarly, if the first roll is a 1, then you have to roll $k-1$ with the single remaining die. The number of ways of doing that is $S(1,k-1)$. Finally, if the first roll is a 0, then you have to roll the full sum of k with the single remaining die. There are $S(1,k)$ ways of doing that. We can use these observations to count the number of ways of rolling two dice and getting a sum of k by partitioning them into four disjoint, exhaustive categories. Like this...

$$S(2,k) = S(2,k-2) + S(1,k-2) + S(1,k-1) + S(1,k)\,.$$

And that is our recurrence relation!

Actually, for the two-dice case, we can do a little better. Recall that if k is even, then $S(1,k) = 2$, and if k is odd, then $S(1,k) = 1$. Thus, for all even values of k, $S(1,k-2) = S(1,k) = 2$ and $S(1,k-1) = 1$. Hence, $S(2,k) = S(2,k-2) + 5$. Similar reasoning shows that, for odd values of k, $S(2,k) = S(2,k-2) + 4$. This implies that

$$S(2,k) = \begin{cases} 2k & \text{if } k \text{ is odd;} \\ 5k/2 & \text{if } k \text{ is even.} \end{cases}$$

The resulting sequence $(2, 5, 6, 10, 10, 15, 14, \ldots)$ is A184418 in the Online Encyclopedia of Integer Sequences [4].

3 The difficult problems

Three Oathsworn dice

Here is where it starts to get a little difficult. Unlike the one- or two-dice cases, once we get to three dice it becomes possible to "fail" and get a sum of zero without getting all zeros on our initial rolls. Remember, two zeroes on our initial roll means that it's a failure and the entire sum is considered zero. Therefore, it is easy to see that $S(3,1) = 0$ and $S(3,2) = 3$ (the possibilities are 011, 101 and 110).

For $k > 2$, we can use the same strategy we used in the two-dice case. Instead of rolling all three dice at once, we will imagine rolling them one at a time. Then, we will count the number of ways of getting a sum of k in the same four disjoint, exhaustive categories that we used before.

Once again the first roll must be a ②, 2, 1 or 0. If the first roll is a 1 or 2, then we get no bonus roll with the first die and we just have to roll a sum of $k-1$ or $k-2$ (respectively) with the other two dice. The number of ways of doing this is counted by $S(2, k-1)$ and $S(2, k-2)$ (respectively).

If we roll a 0 with the first die, then we have to roll a sum of k with the other two dice, *but we cannot roll another zero or else it's a failure!* The number of ways of doing this is $S(2, k)$ minus the number of ways of rolling a sum of k with two dice in which one of the initial rolls is a 0. There are two possible locations for the zero, and the remaining die must be able to get the sum of k all by itself. So if the first roll is a 0, then the number of ways of rolling k with all three dice is $S(2, k) - 2S(1, k)$.

What if the first roll is a ②? Then we get to roll it again. We now have to roll a sum of $k-2$ with all three dice, but with one exception to the rules: *we can get two zeros as long as one of them is with the first die* (since the next time we roll it will be a bonus roll, and bonus rolls can't cause our attempt to fail). The number of ways of rolling a sum of $k-2$ with all three dice when the first die is a ② is therefore equal to $S(3, k-2)$, plus the number of ways of rolling $k-2$ with two dice in which one of them has an initial roll of 0. There are 2 possible locations for the zero, and then the entire sum of $k-2$ must then be obtained with the sole remaining die. Hence, when the first roll is a ②, the number of ways of rolling k is given by $S(3, k-2) + 2S(1, k-2)$.

Combining all of this gives us the following recursive relation:

$$S(3, k) = S(3, k-2) + 2S(1, k-2) + S(2, k-2) + S(2, k-1) + S(2, k) - 2S(1, k)\,.$$

Note that k and $k-2$ are either both odd or both even, so $S(1, k-2) = S(1, k)$ and those terms cancel. We are left with the following recurrence relation for $k > 2$:

$$S(3, k) = S(3, k-2) + S(2, k-2) + S(2, k-1) + S(2, k)\,.$$

The sequence we get here $(0, 3, 13, 24, 39, 59, 78, \ldots)$ does not match anything currently listed on the Online Encyclopedia of Integer Sequences.

More than three Oathsworn dice

We're now ready for the general problem of finding the value of $S(n, k)$ for $n > 3$. This is more challenging than the three dice problem, but our experience with the previous cases will make this easier. We can start with our familiar strategy of counting in four disjoint, exhaustive categories. First, we need some initial conditions.

Regardless of the value of n, because two zeros on initial rolls results in a failure, the smallest positive sum we'll be able to achieve will be $n-1$ and this can be done in n ways. To see why, note that there are n ways to choose one of the dice to be a 0, and the rest must all be 1s. Hence, $S(n, n-1) = n$.

What about $S(n, n)$? To roll a sum of n with n dice, we must do one of the following:

(a) Roll all 1s;

(b) Roll one 2, one 0 and $n-2$ 1s;

(c) Roll one ②, one 0, $n-2$ 1s, and then a 0 on our bonus roll.

There is clearly just one way to roll all 1s. The number of ways of getting the second kind of outcome is $n(n-1)$ (there are n ways to choose one of the dice to be the 2, and for each such choice there are $n-1$ ways to choose one of the remaining die to be the 0, and the rest 1s). By a similar argument, the number of ways of getting the third kind of outcome is also $n(n-1)$. Thus, $S(n, n) = 1 + 2n(n-1)$. This should take care of our initial conditions.

Based on our prior experience with 3 dice, we must define two more functions. For positive integers n and k, we define $OZ(n, k)$ to be the number of ways of rolling n dice and getting a sum of k with exactly one zero on our initial rolls. We define $NZ(n, k)$ to be the number of ways of rolling n dice and getting a sum of k with no zeros on our initial rolls. Since two or more zeros on initial rolls results in an automatic failure, it should be clear that $S(n, k) = OZ(n, k) + NZ(n, k)$.

Now we're ready to put together our recursive relation for $S(n, k)$ when $k > n$. Once again, we imagine rolling the dice one at a time. The first roll will have to be a ②, 2, 1 or 0. If it's a 1 or a 2, then we don't roll that die again and we have to roll $k-1$ or $k-2$ (respectively) with the remaining $n-1$ dice. The number of ways of doing that is given by $S(n-1, k-1)$ and $S(n-1, k-2)$ (respectively).

What if we roll a 0 with the first die? Then we don't roll that die again, and we have to roll a sum of k with the remaining $n-1$ dice, *but we cannot roll another zero on an initial roll*. The number of ways of accomplishing this is given by $NZ(n-1, k)$ or $S(n-1, k) - OZ(n-1, k)$.

Finally, what if we roll a ② with the first die? We then get to roll that die again, and we need to roll a sum of $k-2$ with all n dice. Moreover, *we can get two zeros as long as we get one of them with the first die (on its bonus roll)*. So the number of ways of rolling a sum of k with n dice when the first roll is a ② is given by $S(n, k-2) + OZ(n-1, k-2)$.

Putting all of this together, we can now construct a recurrence relation that will allow us to compute values of $S(n, k)$ when $n > 3$.

$$\begin{aligned} S(n,k) = S(n,k-2) &+ OZ(n-1,k-2) + S(n-1,k-2) \\ &+ S(n-1,k-1) + S(n-1,k) - OZ(n-1,k)\,. \end{aligned} \tag{1}$$

We have only to figure out how to count $OZ(n-1, k-2)$ and $OZ(n-1, k)$ and we will have all we need!

4 Rolling sums with a single zero

To complete the recurrence relation, we just need to be able to count the number of ways of rolling a given sum when we get a zero among our initial rolls. Consider the case when we have n dice, are trying to roll a sum of k, and we get one zero. It's pretty clear that $OZ(n, k) = 0$ when $n = 1$, but what if $n > 1$?

There are n ways to choose one of the dice to be the zero. For each such choice, there are $NZ(n-1, k)$ ways to roll the sum of k with the remaining $n-1$ dice without getting another zero on an initial roll). Hence, the number of ways of rolling a sum of

k with n dice in which one of the initial rolls is a zero is

$$OZ(n,k) = nNZ(n-1,k)\,. \tag{2}$$

Recall that $NZ(n,k) = S(n,k) - OZ(n,k)$, so for sufficiently large n we can say $NZ(n-1,k) = S(n-1,k) - OZ(n-1,k)$ and equation (2) becomes

$$OZ(n,k) = nS(n-1,k) - nOZ(n-1,k)\,. \tag{3}$$

Now let's repeat this process with $OZ(n-1,k)$. Note that equation (3) implies

$$OZ(n-1,k) = (n-1)S(n-2,k) - (n-1)OZ(n-2,k)\,.$$

Substituting this into equation (3) and simplifying now gives us

$$OZ(n,k) = nS(n-1,k) - n(n-1)S(n-2,k) + n(n-1)OZ(n-2,k)$$

A pattern begins to emerge! If we keep repeating this process, we eventually obtain

$$\begin{aligned} OZ(n,k) = {} & nS(n-1,k) - n(n-1)S(n-2,k) + n(n-1)(n-2)S(n-3,k) \\ & + \cdots + (-1)^n n! S(1,k)\,. \end{aligned}$$

More compactly,

$$OZ(n,k) = \sum_{i=1}^{n-1} (-1)^{i+1} \frac{n!}{(n-i)!} S(n-i,k)\,. \tag{4}$$

Note that (4) can also be established by using something called *the Principle of Inclusion and Exclusion*, but the method shown above may be a bit more intuitive if you're not familiar with that rule. We can now use formulas (1) and (4) to calculate values of $S(n,k)$ and $OZ(n,k)$ when $n > 3$. Several values can be found in Tables 1 and 2.

Table 1: Values of $S(n,k)$

$k \backslash n$	1	2	3	4	5	6	7	8	9	10
0 (fail)	1	1	10	67	376	1909	9094	41479	183412	792697
1	1	2	0	0	0	0	0	0	0	0
2	2	5	3	0	0	0	0	0	0	0
3	1	6	13	4	0	0	0	0	0	0
4	2	10	24	25	5	0	0	0	0	0
5	1	10	39	68	41	6	0	0	0	0
6	2	15	59	132	150	61	7	0	0	0
7	1	14	78	232	365	282	85	8	0	0
8	2	20	108	362	740	846	476	113	9	0
9	1	18	130	536	1335	2002	1715	744	145	10
10	2	25	171	756	2202	4113	4725	3144	1098	181

Table 2: Values of $OZ(n,k)$

$k\backslash n$	1	2	3	4	5	6	7	8	9	10
1	0	2	0	0	0	0	0	0	0	0
2	0	4	3	0	0	0	0	0	0	0
3	0	2	12	4	0	0	0	0	0	0
4	0	4	18	24	5	0	0	0	0	0
5	0	2	24	60	40	6	0	0	0	0
6	0	4	33	104	140	60	7	0	0	0
7	0	2	36	168	320	270	84	8	0	0
8	0	4	48	240	610	780	462	112	9	0
9	0	2	48	328	1040	1770	1624	728	144	10
10	0	4	63	432	1620	3492	4347	3024	1080	180

5 Rolling dice and failing

We have seen how to calculate the number of ways to roll n dice and get a sum of k. We may as well count the number of ways to roll n dice and "fail" by rolling at least two zeros. It turns out we can find a nice formula for this.

For $n = 1$ and $n = 2$, it is obvious. There is just one way for each case (0 and 00, respectively). However, for $n \geq 3$, it is a little more complicated. Once again, we will count the number of ways by partitioning them into disjoint, exhaustive categories: the number of ways of rolling exactly two zeros, plus the number of ways of rolling exactly three zeros, plus the number of ways of rolling exactly four zeros ... and so forth.

To start, let's say we are rolling n dice and want to count the number of ways of getting exactly two zeros. Out of these dice, we choose two of them to be zero. This can be done in $\binom{n}{2}$ ways. Of the remaining $n-2$ dice, each can be a 1, 2 or ② (3 possibilities). Therefore, the number of ways to roll exactly two zeros is given by

$$\binom{n}{2}3^{n-2}\,.$$

Similar calculations can be done to count the number of ways of rolling n dice and getting exactly m zeros for each m from $m = 2$ to $m = n$. For each such m, this is given by

$$\binom{n}{m}3^{n-m}\,.$$

Therefore, the number of ways of rolling at least two zeros when you roll n dice is

$$\binom{n}{2}3^{n-2} + \binom{n}{3}3^{n-3} + \cdots + \binom{n}{n}3^{n-n} = \sum_{m=2}^{n}\binom{n}{m}3^{n-m}\,.$$

Now we're going to apply a common little trick in combinatorics. Note that if we multiply each term by an appropriate power of 1, then our answer starts to look a little bit like the binomial theorem:

$$\sum_{m=2}^{n} \binom{n}{m} 3^{n-m} = \sum_{m=2}^{n} \binom{n}{m} 1^m 3^{n-m} \,.$$

If we add and subtract the missing terms, then we can apply the binomial theorem as follows:

$$\sum_{m=2}^{n} \binom{n}{m} 3^{n-m} = \sum_{m=0}^{n} \binom{n}{m} 1^m 3^{n-m} - \binom{n}{1} 3^{n-1} - \binom{n}{0} 3^n = (1+3)^n - n3^{n-1} - 3^n \,.$$

Therefore, for $n \geq 3$ the number of ways to roll n dice and fail by getting at least two zeros is given by $4^n - n3^{n-1} - 3^n$. Several of these values can be found in the first row of Table 1. They appear to match all but the first couple terms of sequence A086443 of the Online Encyclopedia of Integer Sequences [4].

References

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