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Constant-recursive equations and polynomial equations

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1 Introduction

In this paper, we explore constant-recursive equations, which often appear in mathematical analysis, combinatorics and differential equations. We consider sequences that have a fixed linear relation between their terms and the previous terms; i.e., sequences that satisfy a formula of the form

$$S_{n+k} = c_0 S_n + c_1 S_{n+1} + c_2 S_{n+2} + \cdots + c_{k-1} S_{n+k-1}\,.$$

We investigate the polynomial structures underlying them, characterize the structure of such recurrences, and explain their relationship to polynomials and solutions to polynomial equations, by analysing their algebraic and structural properties.

2 Definitions and examples

We start by introducing the definition of constant-recursive equations and considering some examples; see also [1].

Definition 1. *Each term in a constant-recursive equation is a fixed linear combination of previous terms. That is, they all satisfy a formula of the form*

$$S_{n+k} = c_0 S_n + c_1 S_{n+1} + c_2 S_{n+2} + \cdots + c_{k-1} S_{n+k-1}\,.$$

for some fixed coefficients c_i, ranging over the same domain as the sequence with $k \in \mathbb{Z}^+$. In this paper we will explore the case where these coefficients are complex numbers. That is, $c_i \in \mathbb{C}$ for all $i = 0, 1, \ldots, k$.

The following are some examples of sequences that satisfy some constant-recursive equations.

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The Fibonacci sequence

The Fibonacci sequence F is generated by adding the two preceding terms to obtain the next, with the starting conditions $F_0 = 1, F_1 = 1$:

$$F_n = 0\,,\ 1\,,\ 1\,,\ 2\,,\ 3\,,\ 5\,,\ 8\,,\ 13\,,\ \ldots\,.$$

Thus, it satisfies the recurrence $F_{n+2} = F_n + F_{n+1}$.

Exponential sequences

Examples of exponential sequences S include

$$S_n = 2^n = 1\,,\ 2\,,\ 4\,,\ 8\,,\ 16\,,\ 32\,,\ \ldots$$

which satisfies the recurrence $S_{n+1} = 2S_n$, and

$$S_n = 2 \cdot (-3)^n = 2\,,\ -6\,,\ 18\,,\ -54\,,\ 162\,,\ \ldots$$

that satisfies the recurrence $S_{n+1} = -3S_n$.

Polynomial sequences

Polynomial sequences are sequences S of the form

$$S_n = c_0 + c_1 n + \cdots + c_k n^k\,,$$

such as

$$S_n = 2n = 2\,,\ 4\,,\ 6\,,\ 8\,,\ 10\,,\ \ldots$$

which satisfies the recurrence $S_{n+2} = 2S_{n+1} - S_n$, and

$$S_n = n^2 = 1\,,\ 4\,,\ 9\,,\ 16\,,\ 25\,,\ \ldots$$

that satisfies the recurrence $S_{n+3} = 3S_{n+2} - 3S_{n+1} + f_n$.

Trigonometric sequences

Trigonometric sequences S_n include

$$S_n = \sin(n) = \sin(1)\,,\ \sin(2)\,,\ \sin(3)\,,\ \sin(4)\,,\ \ldots$$

and

$$S_n = \cos(n) = \cos(1)\,,\ \cos(2)\,,\ \cos(3)\,,\ \cos(4)\,,\ \ldots$$

where both sequences satisfy the recurrence

$$S_{n+2} = 2\cos(1) S_{n+1} - S_n\,.$$

For a given recurrence, is it possible to find more than one solution? Consider the recurrence $S_{n+1} = 2S_n$, where the trivial solution would be $S_n = 2^n$. A closer look reveals that $S_n = -2^n$ and $S_n = 3 \cdot 2^n$ also work. Furthermore, we can conclude that $S_n = c\,2^n$, for any constant c, satisfies the given recurrence. Hence, we can find more than one solution to the same recurrence. Indeed, the general solution to the degree-1 constant-recursive recurrences is as follows.

Lemma 2. *A sequence S satisfies $S_{n+1} - aS_n = 0$ if and only if $S_n = S_0\,a^n$.*

Proof. Suppose that $S_{n+1} - aS_n = 0$ for all integers $n \geq 0$ and note that $S_1 = aS_0$. Assume that $S_n = S_0\,a^n$ for some integer $n \geq 0$. Then

$$S_{n+1} = aS_n = a(S_0\,a^n) = S_0\,a^{n+1}\,.$$

By induction, $S_n = S_0\,a^n$ for all integers $n \geq 0$.

Conversely, the sequence $S_n = S_0\,a^n$ satisfies $S_{n+1} - aS_n = S_0\,a^{n+1} - a\,S_0\,a^n = 0$. $\square$

3 Transition to operators

Now, let's consider the sequences that satisfy $S_{n+2} - 5S_{n+1} + 6S_n = 0$. We have seen that exponentials are related to these recurrences, so assume that $S_n = c\,a^n$ for some $c,\ a \neq 0$. The equation then becomes

$$c\,a^{n+2} - 5c\,a^{n+1} + 6c\,a^n = 0\,,$$

so $c\,a^n\,(a^2 - 5a + 6) = 0$. Since c and a^n are non-zero, $a^2 - 5a + 6 = 0$, from which it follows that a is either 2 or 3. Therefore, either $S_n = c_0\,2^n$ or $S_n = c_1\,3^n$ for some constants c_0 and c_1. Furthermore, their sum $S_n = c_0\,2^n + c_1\,3^n$ is also a solution.

However, are these the only sequences that work? To answer that question, we will have to shift our perspective and view

$$S_{n+2} - 5S_{n+1} + 6S_n$$

and other recurrences as operators. This will help us understand which sequences vanish under these operators and establish the family of solutions for such recurrences.

Constant-coefficient linear recurrence operators

A *constant-coefficient linear recurrence operator* acting on some sequence S, is a function on S defined by

$$T[S]_n = c_0S_n + c_1S_{n+1} + \cdots + c_kS_{n+k}$$

for some fixed coefficients $c_i \in \mathbb{C}$, where $c_k \neq 0$ and $k \in \mathbb{Z}^+$ is the *degree* of T. Additionally, call $P(x) = c_0 + c_1x + \cdots + c_kx^k$ the *characteristic polynomial*.

Some examples are given below.

The forward difference Δ operator

The forward difference Δ operator is defined by

$$\Delta[S]_n = S_{n+1} - S_n\,,$$

which can be thought of as the "discrete" version of the first derivative operator

$$Df(x) = \frac{d}{dx}f(x)\,.$$

The Fibonacci operator

This operator T is defined by

$$T[S]_n = S_{n+2} - S_{n+1} - S_n\,,$$

and evaluates to 0 whenever S is the Fibonacci sequence.

Operator related to polynomials

The operator T defined by

$$T[S]_n = S_{n+4} - 4S_{n+3} + 6S_{n+2} - 4S_{n+1} + S_n$$

evaluates to 0 whenever S is a polynomial sequence of degree 3 or less.

Properties of constant-coefficient linear recurrence operators

Let us explore the structure of constant-coefficient linear recurrence operators. To do this, suppose that $T_1[S]_n = S_{n+1} - 3S_n$ and $T_2[S]_n = S_{n+1} - 2S_n$. Observe that

$$\begin{aligned} T_1 \circ T_2[S]_n &= T_1\left[T_2[S]_n\right] \\ &= T_1\left[S_{n+1} - 2S_n\right] \\ &= \left(S_{n+2} - 2S_{n+1}\right) - 3\left(S_{n+1} - 2S_n\right) \\ &= S_{n+2} - 5S_{n+1} + 6S_n\,. \end{aligned}$$

and

$$\begin{aligned} T_2 \circ T_1[S]_n &= T_2\left[T_1[S]_n\right] \\ &= T_2\left[S_{n+1} - 3S_n\right] \\ &= \left(S_{n+2} - 3S_{n+1}\right) - 2\left(S_{n+1} - 3S_n\right) \\ &= S_{n+2} - 5S_{n+1} + 6S_n\,. \end{aligned}$$

Therefore, $T_1 \circ T_2 = T_2 \circ T_1$. Hence, we hypothesize that the composition of constant-coefficient linear recurrence operators is commutative.

On the other hand, when we expand $(x-3)(x-2) = x^2 - 5x + 6$, it is clear that the expansion of $(S_{n+1} - 3S_n) \circ (S_{n+1} - 2S_n)\,[S]_n$ works in the exact same way as the expansion of $(x-3)(x-2)$.

Recall our previous discussion on the recurrence $S_{n+2} - 5S_{n+1} + 6S_n = 0$. Now, recognising that

$$S_{n+2} - 5S_{n+1} + 6S_n = (S_{n+1} - 3S_n) \circ (S_{n+1} - 2S_n)$$

makes it clear that $S_n = c_0\, 2^n$ and $S_n = c_1\, 3^n$ being solutions is not a coincidence, since $S_n = c_0\, 2^n$ is a solution to $S_{n+1} - 2S_n = 0$, and $S_n = c_1\, 3^n$ is a solution to $S_{n+1} - 3S_n = 0$.

Let us prove some properties of constant-coefficient linear recurrence operators.

Linearity

Lemma 3. *Constant-coefficient linear recurrence operators T are linear. In other words,*

(a) $T[S+W]_n = T[S]_n + T[W]_n$

(b) $T[aS]_n = aT[S]_n$ *for all* $a \in \mathbb{C}$.

Proof. Let $T[S]_n = c_0 S_n + c_1 S_{n+1} + \cdots + c_k S_{n+k}$. Then

$$\begin{aligned} T[S+W]_n &= c_0\,(S_n + W_n) + c_1\,(S_{n+1} + W_{n+1}) + \cdots + c_k\,(S_{n+k} + W_{n+k}) \\ &= (c_0 S_n + c_1 S_{n+1} + \cdots + c_k S_{n+k}) + (c_0 W_n + c_1 W_{n+1} + \cdots + c_k W_{n+k}) \\ &= T[S]_n + T[W]_n \end{aligned}$$

and
$$\begin{aligned} T[aS]_n &= c_0 a S_n + c_1 a S_{n+1} + \cdots + c_k a S_{n+k} \\ &= a\,(c_0 S_n + c_1 S_{n+1} + \cdots + c_k S_{n+k}) \\ &= aT[S]_n\,. \end{aligned}$$
□

The polynomial nature of operators

The composition of constant-coefficient linear recurrence operators is identical to polynomial multiplication of the characteristic polynomials, as the lemma below describes.

Lemma 4. *Let operators T_1 and T_2 be given by*

$$\begin{aligned} T_1[S]_n &= a_0 S_n + a_1 S_{n+1} + \cdots + a_k S_{n+k} \\ T_2[S]_n &= b_0 S_n + b_1 S_{n+1} + \cdots + b_m S_{n+m}\,. \end{aligned}$$

and denote their characteristic polynomials respectively as

$$\begin{aligned} P_1(x) &= a_0 + a_1 x + \cdots + a_k x^k \\ P_2(x) &= b_0 + b_1 x + \cdots + b_m x^m\,. \end{aligned}$$

Then $T_1 \circ T_2[S]_n = c_0 S_n + c_1 S_{n+1} + \cdots + c_{k+m} S_{n+(m+k)}$
if and only if $P_1(x)P_2(x) = c_0 + c_1 x + \cdots + c_{k+m} x^{k+m}$.

Thus, composition of operators is commutative, associative and distributive with respect to addition, carrying all properties of polynomial multiplication.

Proof. Note that

$$T_1 \circ T_2[S]_n = T_1\left[T_2[S]_n\right]_n = T_1\left[b_0 S_n + b_1 S_{n+1} + \cdots + b_m S_{n+m}\right]_n \;.$$

By the linearity of T_1 given by Lemma 3 and the definition of T_1,

$$T_1 \circ T_2[S]_n = T_1\left[T_2[S]_n\right]_n = \sum_{\ell=0}^{m} b_\ell T_1[S_{n+\ell}]_n = \sum_{\ell=0}^{m} b_\ell \sum_{u=0}^{k} a_u S_{n+\ell+u} = \sum_{\ell=0}^{m}\sum_{u=0}^{k} a_u b_\ell S_{n+\ell+u} \,.$$

By grouping terms containing the same S_{n+v} and re-indexing by $v = \ell + u$, we get

$$T_1 \circ T_2[S]_n = \sum_{v=0}^{k+m}\left(\sum_{u=0}^{v} a_u b_{v-u}\right) S_{n+v} = \sum_{v=0}^{k+m} c_v S_{n+v} \qquad \text{where} \qquad c_v = \sum_{u=0}^{v} a_u b_{v-u} \,.$$

Similarly,

$$P_1(x)P_2(x) = \left(\sum_{u=0}^{k} a_u x^u\right)\left(\sum_{\ell=0}^{m} b_\ell x^\ell\right) = \sum_{u=0}^{k}\sum_{\ell=0}^{m} a_u b_\ell x^{u+\ell} = \sum_{v=0}^{k+m}\left(\sum_{u=0}^{v} a_u b_{v-u}\right) x^v \,.$$

Therefore,

$$P_1(x)P_2(x) = \sum_{v=0}^{k+m} c_v x^v \qquad \text{where} \qquad c_v = \sum_{u=0}^{v} a_u b_{v-u} \,.$$

Thus, we see that the coefficients of S_{n+v} in the composition match the coefficients of x^v in the characteristic polynomial product.

The last part of the theorem follows directly, and its proof is left to the reader. □

4 General solutions to constant-recursive equations

In the next lemma, we show that an analogue of the Fundamental Theorem of Algebra holds in the context of constant-coefficient linear recurrence operators. This result is central to exploring the general solution of constant-recursive equations.

Lemma 5. *Every operator can be factored into a composition of degree-1 operators. Specifically, every $T[S]_n$ has a representation*

$$T[S]_n = (cS_n) \circ (S_{n+1} - Z_0 S_n)^{r_0} \circ (S_{n+1} - Z_1 S_n)^{r_1} \circ \cdots \circ (S_{n+1} - Z_j S_n)^{r_j}\, [S]_n$$

for distinct $Z_i \in \mathbb{C}$, where the powers $r_i \in \mathbb{Z}^+$ represent repeated composition.

Proof. By the Fundamental Theorem of Algebra, we know that the characteristic polynomial can be written as

$$c_0 + c_1 x + \cdots + c_k x^k = c_k\,(x - Z_0)^{r_0}\,(x - Z_1)^{r_1} \cdots (x - Z_j)^{r_j}$$

where $Z_i \in \mathbb{C}$ and $r_i \in Z^+$. Then, by Lemma 5, we conclude that

$$\begin{aligned} T[S]_n &= c_0 S_n + c_1 S_{n+1} + \cdots + c_k S_{n+k} \\ &= (c_k S_n) \circ (S_{n+1} - Z_0 S_n)^{r_0} \circ (S_{n+1} - Z_1 S_n)^{r_1} \circ \cdots \circ (S_{n+1} - Z_j S_n)^{r_j} \;. \end{aligned}$$ □

Special sequences

Another key property of constant-coefficient linear recurrence operators is that they behave in a predictable and nice way on sequences of the form

$$M_n = p_1(n)Z_1^n + p_2(n)Z_2^n + \cdots + p_j(n)Z_j^n$$

where $p_1, p_2, \ldots, p_j$ are polynomials with complex coefficients. Let's state what we exactly mean by "predictable and nice" and prove this property. We will see later that all general solutions take this exact form.

Lemma 6. *Every sequence of the form*

$$M_n = p_1(n)Z_1^n + \cdots + p_j(n)Z_j^n\ ,$$

where each p_i is a polynomial with complex coefficients, lies in the image of every operator. More precisely, for any operator

$$T[S]_n = c_0 S_n + c_1 S_{n+1} + \cdots + c_k S_{n+k}$$

and sequence M_n as above, there exists another sequence

$$W_n = q_1(n)Z_1^n + \cdots + q_j(n)Z_j^n\ ,$$

where $\deg(p_i) = \deg(q_i)$ for all $1 \leq i \leq j$, such that

$$T[W]_n = M_n\ .$$

Proof. We use induction. For $k = 0$, the lemma is true: with $T[S]_n = c_0 S_n$, where $c_0 \neq 0$, and $M_n = p_1(n)Z_1^n + \cdots + p_j(n)Z_j^n$, we see that $W_n = \frac{M_n}{c_0}$ satisfies $T[W]_n = c_0 \frac{M_n}{c_0} = M_n$.

Now assume that the statement holds for all $i < k$ for any $k > 0$. By Lemma 5,

$$\begin{aligned} T[S]_n &= (cS_n) \circ (S_{n+1} - Z_0 S_n)^{r_0} \circ (S_{n+1} - Z_1 S_n)^{r_1} \circ \cdots \circ (S_{n+1} - Z_j S_n)^{r_j} [S]_n \\ &= T_1 \circ T_2[S]_n\ , \end{aligned}$$

where

$$\begin{aligned} & T_1[S] = (S_{n+1} - Z_0 S_n)^{r_0} [S]_n \\ \text{and} \qquad & T_2[S] = (S_{n+1} - Z_0 S_n)^{r_0} \circ (S_{n+1} - Z_1 S_n)^{r_1} \circ \cdots \circ (S_{n+1} - Z_j S_n)^{r_j} [S]_n \end{aligned}$$

for some distinct $Z_i \in \mathbb{C}$ and positive integers $r_i \in \mathbb{Z}^+$. Since $r_0 < k$, the induction assumption implies that there exists $W_n = q_1(n)Z_1^n + \cdots + q_j(n)Z_j^n$ so that $T_1[W]_n = M_n$, where $\deg(p_i) = \deg(q_i)$ for all $1 \leq i \leq j$. Similarly, since $k - r_0 < k$, there exists $B_n = g_1(n)Z_1^n + \cdots + g_j(n)Z_j^n$ such that $T_2\left[B\right]_n = W_n$, where $\deg\left(q_i\right) = \deg(g_i)$ for all $1 \leq i \leq j$. Therefore,

$$T[S]_n = T_1\left[T_2\left[B\right]_n\right]_n = T_1[W]_n = M_n\ ,$$

where $\deg\left(p_i\right) = \deg\left(g_i\right)$ for all $1 \leq i \leq j$. Induction concludes the proof. $\square$

General solutions

The general solution of constant-recursive equations is given by the theorem below.

Theorem 7. *A sequence S_n satisfies*

$$c_0 S_n + c_1 S_{n+1} + \cdots + c_k S_{n+k} = 0$$

if and only if

$$S_n = p_0(n) Z_0^n + p_1(n) Z_1^n + \cdots + p_j(n) Z_j^n$$

where $c_0 + c_1 Z_i + \cdots + c_k Z_i^k = 0$ and p_i are polynomials with degrees less than the multiplicity of the root Z_i in the characteristic polynomial of the recurrence.

Proof. Define the operator $Q[S]_n = c_0 S_n + c_1 S_{n+1} + \cdots + c_k S_{n+k}$. By Lemma 5,

$$Q[S]_n = (cS_n) \circ (S_{n+1} - Z_0 S_n)^{r_0} \circ (S_{n+1} - Z_1 S_n)^{r_1} \circ \cdots \circ (S_{n+1} - Z_j S_n)^{r_j} [S]_n$$

for distinct Z_i.

Suppose that $S_n = p_0(n) Z_0^n + p_1(n) Z_1^n + \cdots + p_j(n) Z_j^n$ as in the theorem. We use induction on k to prove that $c_0 S_n + c_1 S_{n+1} + \cdots + c_k S_{n+k} = 0$. Certainly, this is true for $k = 0$: then $c_0 = 0$ and so $c_0 S_n = 0$.

Assume the statement holds for all $i < k$. By Lemma 3,

$$\begin{aligned} &Q\left[p_0(n) Z_0^n + p_1(n) Z_1^n + \cdots + p_i(n) Z_j^n\right]_n \\ &= Q\left[p_0(n) Z_0^n\right]_n + Q\left[p_1(n) Z_1^n\right]_n + \cdots + Q\left[p_i(n) Z_j^n\right]_n . \end{aligned}$$

Since $r_i < k$, the induction assumption implies that $(S_{n+1} - Z_i S_n)^{r_i} [p_i(n) Z_i^n]_n = 0$ for all $0 \le i \le j$. Therefore, $Q[p_i(n) Z_i^n]_n = 0$ for every $0 \le i \le j$, and so

$$c_0 S_n + c_1 S_{n+1} + \cdots + c_k S_{n+k} = Q\left[p_0(n) Z_0^n\right]_n + Q\left[p_1(n) Z_1^n\right]_n + \cdots + Q\left[p_i(n) Z_j^n\right]_n = 0 .$$

Now suppose that $c_0 S_n + c_1 S_{n+1} + \cdots + c_k S_{n+k}$. We use induction on k to prove that S_n is given as in the theorem. Certainly, this is true for $k = 0$: then $c_0 S_n = 0$ and so $S_n = 0$ since $c_0 \neq 0$.

Assume the statement holds for all $i < k$. Suppose that W_n satisfies

$$Q[W]_n = c_0 W_n + c_1 W_{n+1} + \cdots + c_k W_{n+k} = 0 .$$

Note that

$$(cS_n) \circ (S_{n+1} - Z_1 S_n)^{r_1} \circ \cdots \circ (S_{n+1} - Z_j S_n)^{r_j} \left[(S_{n+1} - Z_0 S_n)^{r_0} [W]_n\right]_n = 0$$

and, since $k - r_0 < k$, the induction assumption implies that

$$(S_{n+1} - Z_0 S_n)^{r_0} [W]_n = q_1(n) Z_1^n + \cdots + q_j(n) Z_j^n ,$$

where $\deg(q_i) < r_i$, for all $1 \le i \le j$. By Lemma 6, there exists a sequence T_n of the form

$$T_n = p_1(n) Z_1^n + \cdots + p_j(n) Z_j^n ,$$

with $\deg(p_i) = \deg(q_i) < r_i$ for all $1 \leq i \leq j$, such that

$$(S_{n+1} - Z_0 S_n)^{r_0} [T]_n = q_1(n) Z_1^n + \cdots + q_j(n) Z_j^n$$

Observe that

$$\begin{aligned}(S_{n+1} - Z_0 S_n)^{r_0} [W - T]_n &= (S_{n+1} - Z_0 S_n)^{r_0} [W]_n - (S_{n+1} - Z_0 S_n)^{r_0} [T]_n \\ &= \left(q_1(n) Z_1^n + \cdots + q_j(n) Z_j^n\right) - \left(q_1(n) Z_1^n + \cdots + q_j(n) Z_j^n\right) = 0\,.\end{aligned}$$

Therefore,

$$(S_{n+1} - Z_0 S_n)^{r_0} [W - T]_n = 0$$

and, since $r_0 < k$, the induction assumption implies that

$$W_n - T_n = p_0(n) Z_0^n$$

with $\deg(p_0) < r_0$. Then

$$W_n = T_n + p_0(n) Z_0^n = p_0(n) Z_0^n + p_1(n) Z_1^n + \cdots + p_j(n) Z_j^n,$$

where $\deg(p_i) < r_i$, for all $0 \leq i \leq j$. Note that r_i equals the multiplicity of the root Z_i. Thus, we have completed the proof. □

Summary and example

In order to find the general solution of a constant-recursive equation, for example

$$S_{n+4} - 24 S_{n+2} + 64 S_{n+1} - 48 S_n = 0,$$

we find the roots for the corresponding characteristic polynomial

$$x^4 - 24x^2 + 64x - 48,$$

which factors as

$$(x-2)^3 (x+6)$$

We have 2 roots, namely $x = 2$ with multiplicity 3 and $x = -6$ which has multiplicity 1. Then, all solutions are the sum over all roots of the characteristic polynomial of terms that has the form $p(n)Z^n$, where $p(n)$ is a polynomial that has a degree less than the multiplicity of the root Z in the characteristic polynomial. In our example, the general solution would be $S_n = c_0(-6)^n + (c_1 + c_2 n + c_3 n^2)\, 2^n$.

5 Conclusion

We found that recurrence operators, like those defining sequences such as the Fibonacci numbers, behave much like polynomials. This paper provides a great example of an algebraic structure called a ring, which captures the main properties of addition and multiplication over the integers (commutativity, associativity and distributivity), and carries them into other sets. The structural equivalence that we have seen between composition of operators and multiplication of polynomials, which forms the basis of the duality that we explored, is an example of an isomorphism, thereby establishing a stronger structural correspondence between the algebraic systems involved. We hope that our explorations in this paper inspire readers to dive into these exciting topics in mathematics.

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