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Analysis of amicable numbers and their abundancy index: patterns, bounds and implications

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1 Introduction

Amicable numbers are two positive integers where each number is the sum of the proper divisors of the other. The abundancy index of a number is the ratio of the sum of all its divisors to the number itself. This article looks at the connection between amicable numbers and the abundancy index. Since it is still unknown whether there are infinitely many amicable numbers, studying patterns in their abundance index might help us to understand them better. Working directly with large amicable numbers is difficult, but the abundancy index, which is a simple ratio, is easier to analyse.

In this research, the abundancy index was calculated for approximately 170,000 known amicable numbers, up to 10^{15}. The distribution of these indices was displayed using a bar graph representing abundancy index versus frequency. Furthermore, formulas due to Thâbit ibn Qurrah — known to produce certain amicable pairs — were examined to determine whether they yield any consistent abundancy index values.

2 Definitions and examples

The *proper divisors* of a positive number n are the positive divisors of n excluding n itself. The sum of all positive divisors of n, including n itself, is denoted by $\sigma(n)$.

Definition 1. *Two positive integers m and n form an* amicable pair *if each number is equal to the sum of the proper divisors of the other. That is, if $\sigma(m) = \sigma(n) = m + n$, then m and n form an amicable pair.*

Example. Let $m = 220$ and $n = 284$. Then

$$\begin{aligned}\sigma(m) &= 1+2+4+5+10+11+20+22+44+55+110+220 = 504 = m+n\,;\\ \sigma(n) &= 1+2+4+71+142+284 = 504 = m+n = \sigma(m)\,.\end{aligned}$$

Hence, 220 and 284 form an amicable pair.

Definition 2. *The* abundancy index *of a positive integer n is $I(n) = \dfrac{\sigma(n)}{n}$.*

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Example. The positive divisors of $n = 12$ are

$$1\,,2\,,3\,,4\,,6\,,12\,,$$

so $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$. Thus, the abundancy index of 12 is

$$I(12) = \frac{\sigma(12)}{12} = \frac{28}{12} = \frac{7}{3} \approx 2.33\,.$$

Definition 3. *A* perfect number n *is a positive integer that is equal to the sum of its proper divisors, that is,* $\sigma(n) = 2n$.

Example. The positive divisors of $n = 28$ are

$$1\,,2\,,4\,,7\,,14\,,28\,,$$

so $\sigma(28) = 1 + 2 + 4 + 7 + 14 + 28 = 56$. Hence, the abundancy index of 28 is

$$I(28) = \frac{\sigma(28)}{28} = \frac{56}{28} = 2\,,$$

and 28 is a perfect number.

3 History of amicable numbers

The earliest mention of amicable numbers dates back to Pythagoras in the 6th century BCE. The first known pair of amicable numbers, $(220, 284)$, was discovered by Thâbit ibn Qurrah [3] in the ninth century, and work on these pairs was conducted by others around that time. Thâbit ibn Qurrah's Theorem provides a way to generate amicable numbers when certain conditions are satisfied:

Theorem 4 (Thâbit ibn Qurrah's Theorem)**.** [3] *If*

$$\begin{aligned} p &= 3 \cdot 2^{k-1} - 1\,, \\ q &= 3 \cdot 2^{k} - 1\,, \\ r &= 9 \cdot 2^{2k-1} - 1 \end{aligned}$$

are each prime numbers for some integer $k > 1$, then $m = 2^k pq$ and $n = 2^k r$ form an amicable pair of numbers.

Amicable numbers regained attention in the 17th and 18th centuries, especially due to work by Euler [4, pp. 49–55] who developed a formula for generating such pairs, based on previous work by Fermat and others. Despite much subsequent work, it is still unknown whether there are infinitely many amicable numbers.

4 Properties of the abundancy index

Below are some of the properties that will be useful later in this article.

Theorem 5. [2] *The abundancy index $I(n)$ is a multiplicative function. That is, $I(mn) = I(m)I(n)$ for all coprime positive integers m and n.*

The above theorem follows from the fact (see [2]) that $\sigma(n)$ is a multiplicative function:

$$I(mn) = \frac{\sigma(mn)}{mn} = \frac{\sigma(m)\sigma(n)}{mn} = \frac{\sigma(m)}{m}\,\frac{\sigma(n)}{n} = I(m)I(n).$$

Theorem 6. [1] *$I(kn) \geq I(n)$ for all $k \in \ltimes$, and equality holds if and only if $k = 1$.*

Theorem 7. *There are infinitely many non-perfect numbers.*

Proof. There are infinitely many prime numbers p and none of them is perfect, since $\sigma(p) = 1 + p < 2p$. □

Note that, for any positive integer n,

$$I(n) = \frac{\sigma(n)}{n} = \sum_{d|n} \frac{1}{d}\,. \tag{1}$$

Theorem 8. [1] *If $p_1, \ldots, p_k$ are the prime factors of a positive integer n, then*

$$\prod_{i=1}^{k} \frac{p_i + 1}{p_i} \leq I(n) \leq \prod_{i=1}^{k} \frac{p_i}{p_i - 1}\,.$$

Proof. Consider p to be a prime and α any positive integer. Then by Equation (1),

$$\frac{p+1}{p} = 1 + \frac{1}{p} \leq I(p^\alpha) = 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^\alpha} < \sum_{i=0}^{\infty} \frac{1}{p^i} = \frac{1}{1-p}\,. \tag{2}$$

Write $n = p_i^{\alpha_1} \cdots p_k^{\alpha_k}$ where each α_i is a positive integer. Since I is multiplicative by Theorem 5,

$$I(n) = \prod_{i=1}^{k} I(p_i^{\alpha_i})\,.$$

Now apply Equation (2). □

5 Results

We now present three results that we have found regarding $I(n)$. The first is the fundamental identity given in the theorem below.

Theorem 9. *If (m,n) is an amicable pair, then $I(m)+I(n)=I(m)I(n)$.*

Proof. Since $\sigma(m)=\sigma(n)=m+n$,

$$I(m)+I(n)=\frac{\sigma(m)}{m}+\frac{\sigma(n)}{n}=\frac{n\sigma(m)+m\sigma(n)}{mn}=\frac{(m+n)^2}{mn}=\frac{\sigma(m)}{m}\,\frac{\sigma(n)}{n}=I(m)I(n)\,.$$

□

The next result provides bounds on the abundancy indices of amicable numbers.

Theorem 10. *If (m,n) is an amicable pair with $m<n$, then*

$$I(m)>2\,,\quad 1<I(n)<2\quad \textit{and}\quad I(m)+I(n)>4\,.$$

Proof. Clearly, $I(n)>1$. Since $\sigma(m)=\sigma(n)$,

$$\begin{aligned} I(m) &= \frac{\sigma(n)}{m}=1+\frac{n}{m}>1+1=2\,,\\ I(n) &= \frac{\sigma(m)}{n}=1+\frac{m}{n}>1+1=2\,. \end{aligned}$$

Finally, by Theorem 9, $I(m)+I(n)=I(m)I(n)\geq 2I(m)>2\cdot 2=4\,.$ □

For the amicable number pairs (m,n) generated by Thâbit ibn Qurrah's Theorem, we find stronger bounds on $I(m)$ and $I(n)$, as follows.

Theorem 11. *If amicable numbers m and n are generated by Thâbit ibn Qurrah's Theorem, then*

$$\frac{220+284}{284}\leq I(n)<2<I(m)\leq\frac{220+284}{220}\,.$$

Proof. Following Thâbit ibn Qurrah's Theorem, write $m=2^kpq$ and $n=2^kr$ where $p=3\cdot 2^{k-1}-1$, $q=3\cdot 2^k-1$ and $r=9\cdot 2^{2k-1}-1$ are primes and $k>1$ is an integer. Note that

$$m=2^kpq=2^k\big(3\cdot 2^{k-1}-1\big)\big(3\cdot 2^k-1\big)<2^k\big(9\cdot 2^{2k-1}-1\big)=2^kr=n\,,$$

so $2<I(m)$ and $I(n)<2$ by Theorem 10.

Now define the function

$$f(x)=\frac{\big(3\cdot 2^{x-1}-1\big)\big(3\cdot 2^x-1\big)}{9\cdot 2^{2x-1}-1}$$

and differentiate to find

$$f'(x)=9\cdot 2^{x-2}\ln(2)\,\frac{9\cdot 2^{2x}-2^{x+3}+2}{\big(9\cdot 2^{2x-1}-1\big)^2}\,.$$

For all $x \geq 2$, $9 \cdot 2^{2x} - 2^{x+3} = 9 \cdot 2^{x-3}\left(2^{x+3} - 1\right) > 0$, $f'(x) > 0$ and so $f(x)$ is strictly increasing. Therefore, $f(2) = \frac{55}{71}$ is the smallest value of $f(x)$ for all $x \geq 2$. Since $k \geq 2$,

$$\frac{n}{m} = \frac{2^n pq}{2^n r} = \frac{pq}{r} = \frac{\left(3 \cdot 2^{k-1} - 1\right)\left(3 \cdot 2^k - 1\right)}{9 \cdot 2^{2k-1} - 1} = f(k) \geq \frac{55}{71}\,.$$

Therefore,

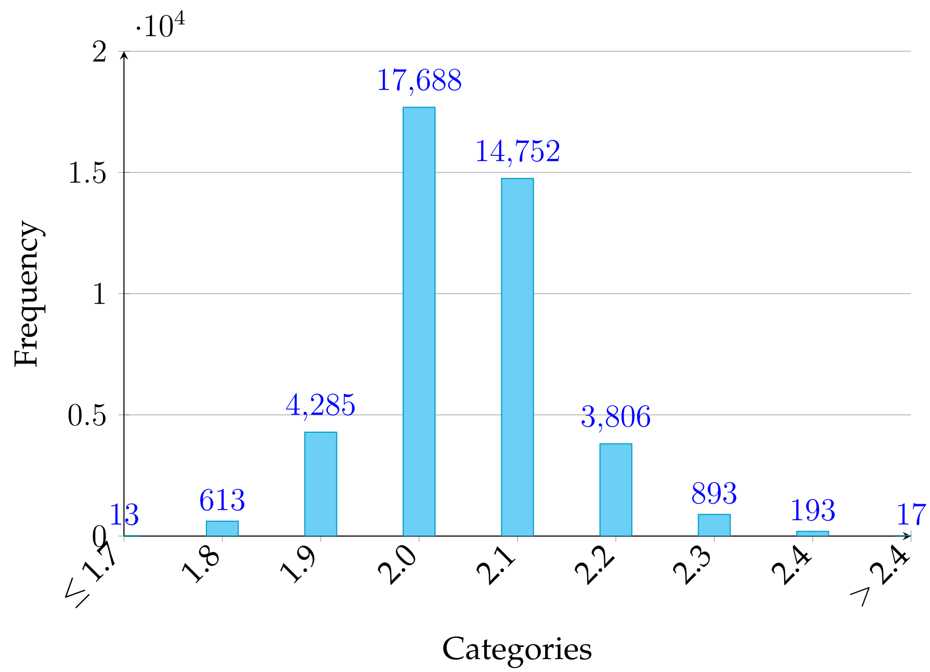
$$I(n) = 1 + \frac{m}{n} \geq 1 + \frac{55}{71} = \frac{126}{71} = \frac{220 + 284}{284}\,.$$

Also, since $\frac{n}{m} \leq \frac{71}{55}$,

$$I(m) = 1 + \frac{n}{m} \leq 1 + \frac{71}{55} = \frac{126}{55} = \frac{220 + 284}{220}\,.$$

$\square$

6 The abundancy index of amicable numbers up to 10^{15}



7 Python code to find the abundancy index of an amicable number

The following Python code calculates the abundancy index of a positive integer.

```
def sigma(n):
    # Function to calculate the sum of divisors of n
    total = 0
    for i in range(1, int(n**0.5) + 1):
        if n % i == 0:
            total += i
            if i != n // i:
                total += n // i
    return total

def abundancy_index(n):
    return sigma(n) / n

# Input from the user
number = int(input("Enter a positive integer: "))
if number <= 0:
    print("Please enter a positive integer.")
else:
    index = abundancy_index(number)
    print(f"Abundancy Index of {number} is {index}")
```

8 Conclusion

This study presents the identity $I(m)I(n) = I(m) + I(n)$ for amicable pairs (m, n), offering a new structural insight. Theoretical bounds and simulations show that their abundancy indices cluster around 2 for amicable numbers up to 10^{15}, or for almost the first 170000 amicable numbers, mostly within the range $[1.7; 2.2]$. For large numbers, abundancy indices below 2 are rare, suggesting a declining density of amicable pairs as their size increases.

9 Future Work

As each perfect number has an abundancy index of exactly two and the highest number of amicable numbers has an abundancy index of around 2, is there any connection between the density of perfect and amicable numbers? Also, are the strong bounds in Theorem 11 true for all, or most, amicable numbers?

These are among the interesting open questions worth investigating in future work.

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References

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[4] E.C. Sandifer, *How Euler Did It*, Mathematical Association of America, 2007.