Parabola Volume 61 Issue 3 (2025), Article 8

Extending factorials to the gamma function

Josephine Miller[1]

1 Introduction

Factorials are fundamental operations in mathematics, and they appear in many fields, from combinatorics to physics. However, the factorial as traditionally defined has a limitation: it is only valid for integers. A natural question arises: Can factorials be applied to non-integer values, such as $(1.5)!$, while maintaining consistency with the traditional definition for integer values?

This question becomes especially relevant in the context of Newton's generalised Binomial Theorem, which generalises the Binomial Theorem to apply to non-integer exponents:

$$(1+x)^k = \sum_{n=0}^{\infty} \frac{k!}{n!(k-n)!} x^n = \sum_{n=0}^{\infty} \binom{k}{n} x^n \, .$$

When k is a positive integer, the binomial expression represents the familiar combinatorial formula. However for non-integers k, the binomial expansion fails because of the limited domain of the factorial function. Therefore, a new method is needed to expand the domain of the factorial function.

Leonhard Euler was the first, in the 1720s, to ask whether a continuous function could satisfy the factorial recurrence relation even for non-integer values. His investigations led to a formula that could compute values such as $(1.5)!$; see [1]. This formula eventually led to the gamma function, which extends the factorial to non-integer values.

Visualizing the factorial function is a natural starting point for looking at extending the domain of the factorial function. A plot of factorial values for $n \in \{0, 1, 2, 3, 4, 5\}$ is shown in Figure 1.

Intuitively, we might start by attempting to estimate fractional factorials by interpolating between the known integer values. There is a neat way to find an interpolation of the factorial function using our familiar calculus tools.

[1]Josephine Miller is a junior student at AEON Institute for Advanced Mathematics.

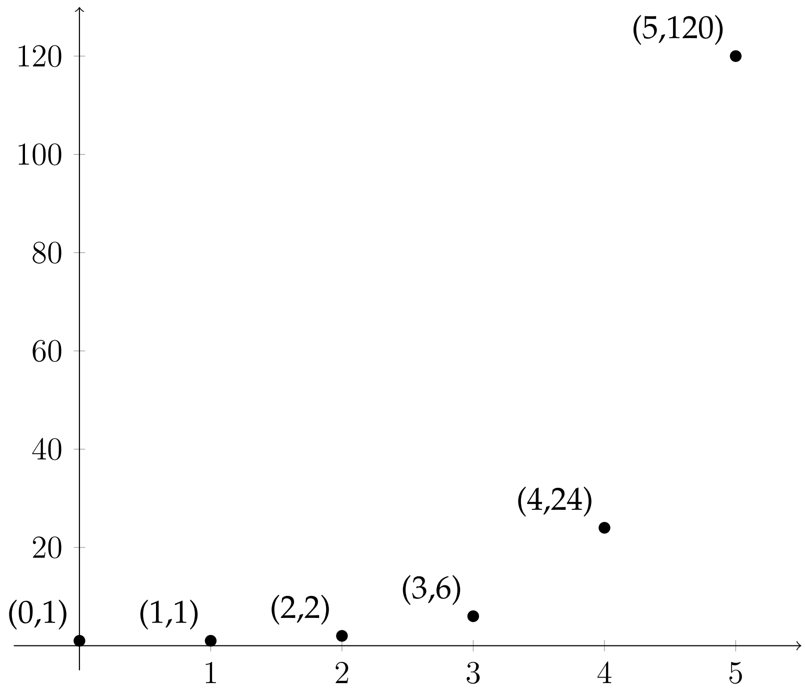

Figure 1: Factorial values for $n \in \{0, 1, 2, 3, 4, 5\}$

2 Derivatives

The derivative is a good starting point, as the power rule is often reminiscent of factorials. Consider a simple polynomial function, $f(x) = x^4$, and observe the emergence of factorials as derivatives are repeatedly found:

$$\frac{d}{dx}\,x^4 = 4 \cdot x^3\,,$$
$$\frac{d^2}{dx^2}\,x^4 = 4 \cdot 3 \cdot x^2\,,$$
$$\frac{d^3}{dx^3}\,x^4 = 4 \cdot 3 \cdot 2 \cdot x\,,$$
$$\frac{d^4}{dx^4}\,x^4 = 4 \cdot 3 \cdot 2 \cdot 1 = 4!\,.$$

The factorial first appears in full at the 4th derivative. Generalising this successive differentiation process to the nth derivative, with $n \in \mathbb{N}$ (where $\mathbb{N}$ denotes the set of natural numbers), we observe that the coefficients take the form $n!$:

$$n! = \frac{d^n}{dx^n}\,x^n\,.$$

Eventually, after taking a sufficiently large number of derivatives, the coefficient will reduce to zero. Using the above example,

$$\frac{d^5}{dx^5}\,x^4 = 0\,,$$

or, more generally,

$$0 = \frac{d^{n+1}}{dx^{n+1}}\, x^n \, .$$

Now, consider a non-polynomial function, $f(x) = x^{-1}$. How might the negative power impact the patterns emerging from the differentiation process in the case of positive integer exponents? Let us expand the derivative to answer this question:

$$\frac{d^n}{dx^n}(x^{-1}) = (-1)^n n(n-1)(n-2)\cdots 3\cdot 2\cdot 1\cdot x^{-(n+1)} = (-1)^n n! x^{-(n+1)} \, .$$

Unlike a polynomial function, after taking a sufficiently large number of derivatives, the coefficient will not reduce to zero. The factorial pattern still appears, but with alternating signs.

Another way of representing $f(x) = \frac{1}{x}$ offers an interesting observation:

$$\int_0^\infty e^{-tx}\, dt = \lim_{w\to\infty} \int_0^w e^{-tx}\, dt = \lim_{w\to\infty}\left(-\frac{1}{x}e^{-xw} - \frac{1}{x}e^{-x(0)}\right) = 0 - \left(-\frac{1}{x}\right) = \frac{1}{x} \, .$$

We can now use this improper integral and repeatedly take derivatives on both sides of the equation, as before:

$$\begin{aligned}
\frac{d}{dx}\int_0^\infty e^{-tx}\, dt &= \int_0^\infty -t\;\; e^{-tx}\, dt = -\,\frac{1}{x^2}\, ; \\
\frac{d^2}{dx^2}\int_0^\infty e^{-tx}\, dt &= \int_0^\infty \;\; t^2\, e^{-tx}\, dt = \quad \frac{2\cdot 1}{x^3}\, ; \\
\frac{d^2}{dx^2}\int_0^\infty e^{-tx}\, dt &= \int_0^\infty -t^3\, e^{-tx}\, dt = -\,\frac{3\cdot 2\cdot 1}{x^4}\, ; \\
\frac{d^2}{dx^2}\int_0^\infty e^{-tx}\, dt &= \int_0^\infty \;\; t^4\, e^{-tx}\, dt = \quad \frac{4\cdot 3\cdot 2\cdot 1}{x^5}\, .
\end{aligned}$$

Now, we can generalise this process for $n \in \mathbb{N}$ successive differentiations:

$$\int_0^\infty (-1)^n t^n e^{-tx}\, dt = (-1)^n \frac{n!}{x^{n+1}} \, ,$$

and so

$$\int_0^\infty t^n e^{-tx}\, dt = \frac{n!}{x^{n+1}} \, .$$

To isolate the factorial, we choose $x = 1$:

$$\int_0^\infty t^n e^{-t} dt = n! \, .$$

By re-indexing $n \to n-1$ for $\int_0^\infty t^n e^{-t} dt$, we arrive at

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt = (n-1)! \, . \tag{1}$$

3 The Bohr-Mollerup Theorem

To extend the factorial function meaningfully to non-integer values, we need clear criteria that ensure the extension behaves consistently. The Bohr-Mollerup Theorem provides this framework, as follows:

Theorem 1 (The Bohr-Mollerup Theorem [2])**.**
There is exactly one function $f(x)$ on the positive real numbers that is logarithmically convex, has value $f(1) = 1$ and satisfies the recurrence relation $f(x+1) = xf(x)$ for all $x > 0$.

The three criteria in the Bohr-Mollerup Theorem are fundamental properties of the factorial function. For instance, $(1)! = 1$ and $(n+1)! = n(n)!$. In our search for functions $f(x)$ that extends the factorial to all positive real numbers, we might therefore look for functions that satisfy these three properties. We show below that *Euler's gamma function*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

from from Equation (1) above does indeed satisfy these properties. By the Bohr-Mollerup Theorem, $\Gamma(x)$ is then the unique function that extends the factorial function in this way.

3.1 Recurrence relation

We can quickly verify that

$$\Gamma(1) = \int_0^\infty t^{1-1} e^{-t} dt = \big[-e^{-t} \big]_0^\infty = 0 - (-1) = 1 \,.$$

Next, we check that $\Gamma(x)$ satisfies the recurrence relation $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$. Using integration by parts, we let $u = t^n$ and see that $du = xt^{x-1}dt$. Then $dv = e^{-t}dt$ and so $v = -e^{-t}$:

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt = \left[-t^x e^{-t}\right]_0^\infty - \int_0^\infty x t^{x-1}(-e^{-t}) dt \,.$$

The term $[-t^x e^{-t}]_0^\infty$ tends to 0 at both limits. We are therefore left with

$$\Gamma(x+1) = x \int_0^\infty t^{x-1} e^{-t} dt = x\Gamma(x) \,,$$

and we have thereby confirmed that the gamma function satisfies the recurrence relation.

3.2 Logarithm convexity of Γ(x)

Although the recurrence relation provides a necessary condition for extending the factorial function, it does not uniquely determine a single function. We introduce the definition of log-convexity to show the final condition of the Bohr-Mollerup Theorem is satisfied.

Definition 2 (Logarithmic convexity [3]). *A function $f(x)$ is* logarithmically convex *on the positive real numbers if and only if, for all $x, y, \alpha, \beta > 0$ with $\alpha + \beta = 1$,*

$$\ln f(\alpha x + \beta y) \leq \alpha \ln f(x) + \beta \ln f(y)\,.$$

Next, we introduce a key tool: the Hölder Inequality for definite integrals.

Theorem 3 (Hölder Inequality [4]).
Let f and g be continuous functions on $[a, b]$ and let $0 \leq p \leq q \leq 1$ such that $p + q = 1$. Then

$$\left(\int_a^b f(t)^p g(t)^q \, dt\right) \leq \left(\int_a^b f(t)\, dt\right)^p \left(\int_a^b g(t)\, dt\right)^q .$$

To demonstrate the logarithmic convexity of $\Gamma(x)$ for $x > 0$, we apply the Hölder Inequality, bearing in mind that $\alpha + \beta = 1$:

$$\begin{aligned}
\Gamma(\alpha x + \beta y) &= \int_0^\infty t^{\alpha x + \beta y - 1} e^{-t}\, dt \\
&= \lim_{b \to \infty} \int_0^b t^{\alpha x + \beta y - 1} e^{-t}\, dt \\
&= \lim_{b \to \infty} \int_0^b \left(t^{\alpha(x-1)} e^{-\alpha t}\right)\left(t^{\beta(y-1)} e^{-\beta t}\right) dt \\
&= \lim_{b \to \infty} \int_0^b \left(t^{x-1} e^{-t}\right)^\alpha \left(t^{y-1} e^{-t}\right)^\beta dt \\
&\leq \lim_{b \to \infty} \left(\int_0^b t^{x-1} e^{-t} dt\right)^\alpha \left(\int_a^b t^{y-1} e^{-t}\, dt\right)^\beta \\
&= \Gamma(x)^\alpha\, \Gamma(y)^\beta\,.
\end{aligned}$$

By taking the natural logarithm of both sides and applying logarithm product and power rules, we conclude that

$$\ln \Gamma \alpha x + \beta y) \leq \ln\left(\Gamma(x)^\alpha\, \Gamma(y)^\beta\right) = \ln \Gamma(x)^\alpha + \ln \Gamma(y)^\beta = \alpha \ln \Gamma(x) + \beta \ln \Gamma(y)\,.$$

Thus, the gamma function is logarithmically convex for $x > 0$. The gamma function is therefore the function uniquely determined by the Bohr-Mollerup Theorem, and we have therefore found the function that extends the factorial function. This is illustrated in Figure 2, in which the gamma function $\Gamma(x + 1)$ is shown alongside the factorial values evaluated at the integer values $n \in \{0, 1, 2, 3, 4, 5\}$.

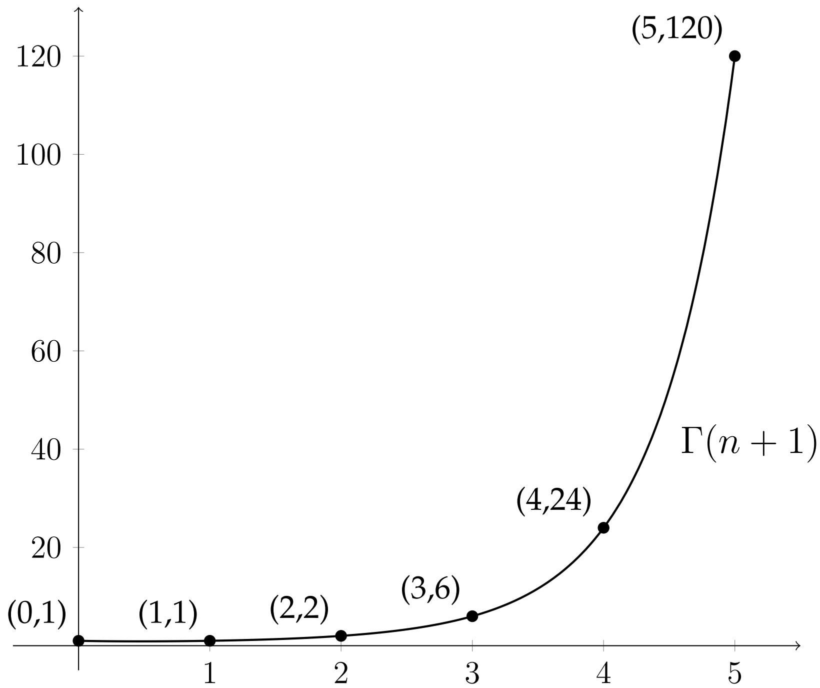

Figure 2: The gamma function $\Gamma(n+1)$ and factorial values for $n \in \{0, 1, 2, 3, 4, 5\}$

4 An example

In our introduction, we posed the question: How do we compute a non-integer factorial such as $(1.5)!$?

Recall the identity that links the gamma function to factorials:

$$n! = \Gamma(n+1)\,.$$

So, to compute $(1.5)!$, we evaluate $(1.5)! = \Gamma(2.5)$. To do this, we can apply the recurrence relation $\Gamma(x+1) = x\Gamma(x)$:

$$(1.5)! = \Gamma(2.5) = 1.5 \cdot \Gamma(1.5) = 1.5 \cdot 0.5 \cdot \Gamma(0.5) = \frac{3}{4}\,\Gamma\Big(\frac{1}{2}\Big)\,.$$

We now derive $\Gamma\big(\frac{1}{2}\big)$, starting from the definition:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{\frac{1}{2}-1} e^{-t}\,dt = \int_0^\infty \frac{e^{-t}}{\sqrt{t}}\,dt\,.$$

Let $y = \sqrt{t}$; then $t = y^2$ and $dt = 2y\,dy$, so

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{e^{-y^2}}{\sqrt{y^2}} \cdot 2y\,dy = 2\int_0^\infty e^{-y^2}\,dy = \int_{-\infty}^\infty e^{-y^2}\,dy\,.$$

We recognize the last integral as the *Gaussian integral*, which is equal to $\sqrt{\pi}$. Therefore,

$$(1.5)! = \Gamma(2.5) = \frac{3}{4}\,\sqrt{\pi}\,.$$

5 Conclusion

In this paper, we have explored extending the factorial function to non-integer values. Starting from motivation rooted in combinatorics and analysis, we recognized a pattern with derivatives to arrive at a representation of the factorial function. We then identified the representation as the gamma function, to which we applied the Bohr-Mollerup Theorem to demonstrate that the gamma function is uniquely determined. Lastly, we applied the result to a question posed in the introduction.

6 Acknowledgements

I sincerely thank Mr. Inaltong and my peers at the Institute for Advanced Mathematics for their guidance, support and invaluable feedback throughout this work.

References

[1] P.J. Davis, Leonhard Euler's integral: A historical profile of the gamma function, *The American Mathematical Monthly* **66** (1959), 849–869.

[2] J. Mollerup and H. Bohr, *Lærebog i Kompleks Analyse*, Vol. 3, Copenhagen, 1922.

[3] Unknown authors, Convexity, logarithmic, *Encyclopedia of Mathematics*, last accessed on 2025-12-23.

[4] Wikipedia, Hölder's inequality, last accessed on 2025-12-23.