

Problems 1781–1790

Parabola would like to thank Randell Heyman for contributing Q1782, and John Winkelman for Q1785.

Q1781 Suppose that n positive integers $a_1, a_2, \dots, a_n$, all less than 2026, satisfy the equation

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} = \frac{2025}{2026}.$$

Find the smallest possible value of n .

Q1782 Let n be a positive integer, and consider the collection of numbers

$$S(n) = \left\{ \left\lfloor \frac{n}{1} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{n}{3} \right\rfloor, \dots, \left\lfloor \frac{n}{n} \right\rfloor \right\}.$$

Here, the notation $\lfloor x \rfloor$ denotes the number x rounded to the nearest integer downwards. For instance,

$$S(10) = \{ 10, 5, 3, 2, 2, 1, 1, 1, 1, 1 \}.$$

Show that the number of elements in $S(n)$, *excluding any repetitions*, is

$$\lfloor \sqrt{4n+1} - 1 \rfloor.$$

Q1783 Let x be a positive integer. Prove that if x^2 is the difference of two consecutive (integer) cubes, then x is the sum of two consecutive (integer) squares.

Q1784 Find the maximum area of a rectangle lying in the first quadrant of the unit circle, if its sides are horizontal and vertical and its bottom left corner is at $(\frac{1}{6}, 0)$.

Q1785 Consider all ten-digit integers formed by arranging the digits $0, 1, \dots, 9$, where an integer may not begin with the digit 0. How many of these integers are multiples of 11?

Q1786 Let

$$f(x) = \frac{x^{100} + 172x^{46} + 1}{x^2 + 1}.$$

Find all positive integers n for which $f(n)$ is an integer.

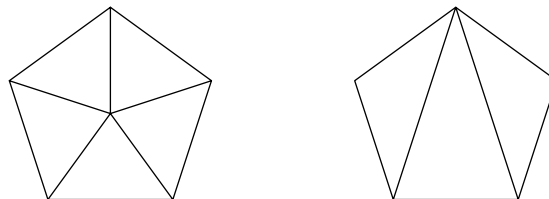
Q1787 A game is played with three dice.

- There is a “selector” die with 6 faces: 5 faces are red and one is blue.
- There is a red die with 20 faces: one face says “WIN” and the others say “LOSE”.
- There is a blue die with 12 faces: one face says “WIN” and the others say “LOSE”.

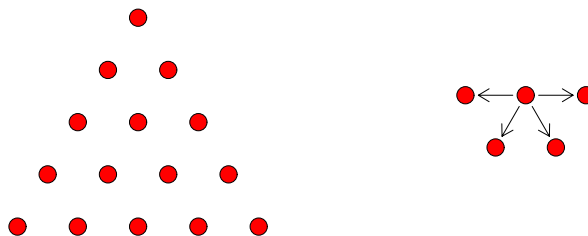
All three dice are rolled. The player wins if and only if either the selector die turns up red and the red die turns up WIN, or the selector die turns up blue and the blue die turns up WIN.

One day, a friend said to me, “I am **so** unlucky! I played this game yesterday, and one of the dice turned up “WIN” (perhaps even both of them – I don’t remember) but I still didn’t win!”. So, question – how unlucky were they really?

Q1788 By considering the following dissections of a regular pentagon, find the exact value of $\cos 36^\circ$.



Q1789 We have a triangular array of dots as shown in the left-hand diagram, extending as far as desired. From any dot, we may move one dot to the left or right or one dot “diagonally downwards”, as shown in the right hand diagram.



We are never allowed to visit the same dot more than once. Beginning in the top row, which we call row 0, how many ways are there to reach row n ?

Q1790 Let A and B be opposite vertices of a cube having side length 1. Find the radius of the sphere which is tangent to each of the three **faces** of the cube which meet at A , and also to the three **edges** of the cube which meet at B .