Parabola Volume 61 Issue 3 (2025), Article 14

Solutions 1771–1780

Q1771 A follow-up to **Q1764**. Consider sequences of positive integers $x_1, \dots, x_{100}$ with the property that either $x_{k+1} = 4x_k$ or $x_k = 5x_{k+1}$ for each $k = 1, \dots, 99$. What is the smallest possible sum $x_1 + \cdots + x_{100}$ for such a sequence?

SOLUTION Let r_k be the ratio x_{k+1}/x_k : each r_k is either 4 or $\frac{1}{5}$. Suppose that 4 occurs m times and $\frac{1}{5}$ occurs $99 - m$ times. If $r_k = 4$ and $r_{k+1} = \frac{1}{5}$, then we have $x_{k+1} = 4x_k$ and $x_{k+2} = \frac{4}{5}x_k$. If we had instead $r_k = \frac{1}{5}$ and $r_{k+1} = 4$, then we should have $x_{k+1} = \frac{1}{5}x_k < 4x_k$ and $x_{k+2} = \frac{4}{5}x_k$; all other x_j would be the same as before; and so the overall sum would be decreased. This shows that to minimise the sum of all x_j, we must take all the $\frac{1}{5}$ ratios first and the 4s afterwards. It is also clear that for a minimal sum, we must choose the smallest of the x_j to be 1; so the minimal sum is

$$5^{99-m} + \cdots + 5^2 + 5 + 1 + 4 + 4^2 + \cdots + 4^m\,.$$

It remains to find the best value of m. If $5^{99-m} > 4^{m+1}$, then we can obtain a smaller sum by removing 5^{99-m} from the beginning of the sequence and adding 4^{m+1} at the end. So, for a minimal sum, we need $5^{99-m} \le 4^{m+1}$, and, for similar reasons, $4^m \le 5^{100-m}$. These inequalities are equivalent to the following inequalities:

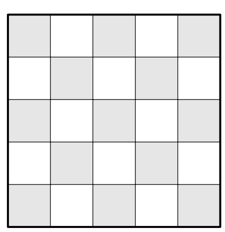
$$\frac{5^{100}}{20} \le 20^m \le 5^{100} \quad \Leftrightarrow \quad \frac{100\log 5}{\log 20} - 1 \le m \le \frac{100\log 5}{\log 20} \quad \Leftrightarrow \quad 52.7 \le m \le 53.7\,,$$

so we take $m = 53$. The minimal sum is

$$5^{46} + 5^{45} + \cdots + 5 + 1 + 4 + 4^2 + \cdots + 4^{53}\,,$$

which works out to $285808535159500622062166407086615$, a number of 33 digits.

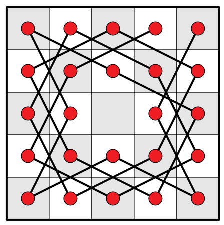
Q1772 In the game of chess, a knight can move from its current square to any square reached by going two squares horizontally or vertically, then one square in a perpendicular direction. There are 13 black squares on a 5×5 chessboard,



and no two of them are separated by a knight's move; moreover, as we showed in the past problem **Q1705** (*Parabola* Volume 59 Issue 1 (2023)), there is no other choice of 13 squares for which this is true.

Now suppose that we want to choose 12 squares on the 5×5 board, such that no two are separated by a knight's move. Clearly we could take all 12 white squares, or any 12 of the 13 black squares. Is there any other way to do this? That is: is it possible to choose 12 squares on a 5×5 chessboard, not all of the same colour, such that no two are separated by a knight's move?

SOLUTION There is only one way to do this. First, note that the central black square on the board cannot be chosen: this would immediately eliminate 8 white squares; choosing any of the remaining allowable white squares would then eliminate 6 black squares, leaving only a maximum of 11 squares to be chosen and making our task impossible. Next, observe that the diagram on the left



18	7	12	1	24
13	2	17	6	11
8	19		23	16
3	14	21	10	5
20	9	4	15	22

shows a path formed of knight's moves which includes all 24 squares on the board, except for the central square. (Regular readers may recall that we used a similar device to solve the earlier problem referenced above.) The diagram on the right is basically the same path, but with squares numbered sequentially for future reference. We may write the path schematically as

$$WBWBWBWBWBWBWBWBWBWBWBWB\,,$$

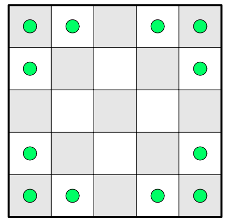
and our aim is to select 12 letters from this sequence, not all the same, without picking two consecutive letters. We may view the desired selection as being made up of 12 letters and 11 gaps, where each gap consists of one or more unused letters.

- If each gap consists of just one letter, then all twelve chosen squares are of the same colour, which is not what we want.
- As there are 12 unused letters altogether and we want 11 gaps, the only remaining option is that 10 gaps consist of just one unused letter, and the other consists of two. This leaves no possibility of unused letters at either end of the sequence, so our selection must include the initial W in square 1 and the final B in square 24.

  Since we must include squares 1 and 24, we must exclude all squares a knight's move away from these; among these are squares 16 and 17, and they must form the gap of two unused letters. Therefore, our choice must be

  $$1\,,\ 3\,,\ 5\,,\ 7\,,\ 9\,,\ 11\,,\ 13\,,\ 15\,,\ 18\,,\ 20\,,\ 22\,,\ 24\,.$$

  This gives the only way to choose 12 squares on the 5×5 chessboard, not all of the same colour, such that no two are separated by a knight's move.



Q1773 We have an unlimited supply of cards which are red on one side, an unlimited supply of green, and one yellow. We create a row of n cards consisting, from left to right, of some number (possibly 0) of green, one yellow and some number (possibly 0) of red. We turn all the cards over so that the colours are unseen, then invite Heidi into the room. She knows all the above information (but does not know how many red and green cards there are). She is allowed to turn over up to 10 cards, one at a time, and she wins if she turns over the yellow card; except that she loses if she turns over four red cards. What is the maximum value of n for which Heidi can be *certain* of winning this game?

Now suppose that there is a row of 100 cards and that Heidi can still, as above, turn over three reds without losing. How many turns does Heidi need to win the game, and how does she do it?

SOLUTION If Heidi is allowed to turn over t cards, and r is the maximum number of red cards she can turn over without losing, then we write $N(t,r)$ for the maximum number of cards such that Heidi can definitely win the game. So, the first problem asks us to determine the value of $N(10,3)$. First, consider the case $t=1$. If Heidi can only turn over one card, then it does not matter how many reds are allowed: she will win if the number of cards is $n=1$, but cannot be certain of winning if $n>1$. Therefore, $N(1,r)=1$ for all $r\geq 0$.

Now consider the situation when $t\geq 2$. If Heidi begins by choosing the mth card from the left, then there are two options that must be considered if she is to guarantee a win:

- If the mth card is red, then Heidi knows that the yellow card must be to its left. As Heidi is now only allowed to turn over $r-1$ red cards and she has used up one of her t attempts, the maximum possible number of cards to the left is $N(t-1,r-1)$.
- If the mth card is green, then the yellow card must be to its right. Heidi is still allowed to turn over r red cards, so the maximum number of cards in this section is $N(t-1,r)$.

Adding up these numbers of cards, and not forgetting to count the mth card as well, shows that

$$N(t,r)=N(t-1,r-1)+1+N(t-1,r)\,. \qquad (*)$$

Now we would like to use this *recursive formula* to find and prove an explicit expression for $N(t,r)$. It should not be surprising that the proof will use the method of mathematical induction; the difficulty with induction is that, essentially, you have to know the answer before beginning the proof. We might also notice that $(*)$ somewhat resembles the well–known "Pascal's triangle" recurrence for the binomial coefficients,

$$\binom{t}{r} = \binom{t-1}{r-1} + \binom{t-1}{r}.$$

If we use $(*)$ to calculate the values of $N(5, r)$ for $r = 0, 1, 2, 3, 4$, then we obtain

$$5\,,\ 15\,,\ 25\,,\ 30\,,\ 31\,,$$

and perhaps we will observe that the difference between successive terms is always a binomial coefficient. The terms can be written as follows:

$$\begin{aligned}
N(5,0) &= \binom{5}{1}, \\
N(5,1) &= \binom{5}{1} + \binom{5}{2}, \\
N(5,2) &= \binom{5}{1} + \binom{5}{2} + \binom{5}{3}, \\
N(5,3) &= \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4}, \\
N(5,4) &= \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}.
\end{aligned}$$

Therefore, we guess that in general,

$$N(t,r) = \binom{t}{1} + \binom{t}{2} + \cdots + \binom{t}{r+1} \tag{$**$}$$

for all $t \geq 1$ and $r \geq 0$. We note that $\binom{t}{k} = 0$ if $k > t$. In the case $t = 1$, Equation $(**)$ says that $N(1, r) = 1$ for all r. This is true, as we have already seen. To proceed with the induction, suppose that $(**)$ is true for $t - 1$; then we can calculate $N(t, r)$ from Equation $(*)$:

$$N(t,r) = \left[\binom{t-1}{1} + \cdots + \binom{t-1}{r}\right] + 1 + \left[\binom{t-1}{1} + \cdots + \binom{t-1}{r+1}\right].$$

Now remember that $1 = \binom{t-1}{0}$; regroup terms; and use the "Pascal's triangle" recurrence to get

$$\begin{aligned}
N(t,r) &= \left[\binom{t-1}{0} + \binom{t-1}{1}\right] + \left[\binom{t-1}{1} + \binom{t-1}{2}\right] + \cdots + \left[\binom{t-1}{r} + \binom{t-1}{r+1}\right] \\
&= \binom{t}{1} + \binom{t}{2} + \cdots + \binom{t}{r+1},
\end{aligned}$$

which confirms that $(**)$ is true for t. By induction, the result is true for all $t \geq 1$. Therefore, the answer to the question asked is

$$N(10,3) = \binom{10}{1} + \binom{10}{2} + \binom{10}{3} + \binom{10}{4} = 10 + 45 + 120 + 210 = 385\,.$$

Comment. If $r \geq t-1$, then we have

$$N(t,r) = \binom{t}{1} + \binom{t}{2} + \cdots + \binom{t}{t} = 2^t - 1\,.$$

This makes sense: if Heidi has t attempts and she is allowed $t-1$ or more red cards, then she will never lose by hitting too many red cards – before this happens she will already have lost by running out of turns. So she can simply do a binary search of up to $2^t - 1$ cards.

To answer the question with a row of 100 cards and three reds allowed, we calculate that $N(7,3) = 98$ and $N(8,3) = 162$. So 7 turns is not quite enough to win the game for sure, but 8 turns is. Since 8 turns could handle a row of 162 cards, but there are only 100, Heidi will have many winning options; one systematic method would be to pretend that an extra 62 red cards are added to the 100. Then from $(*)$ we note that

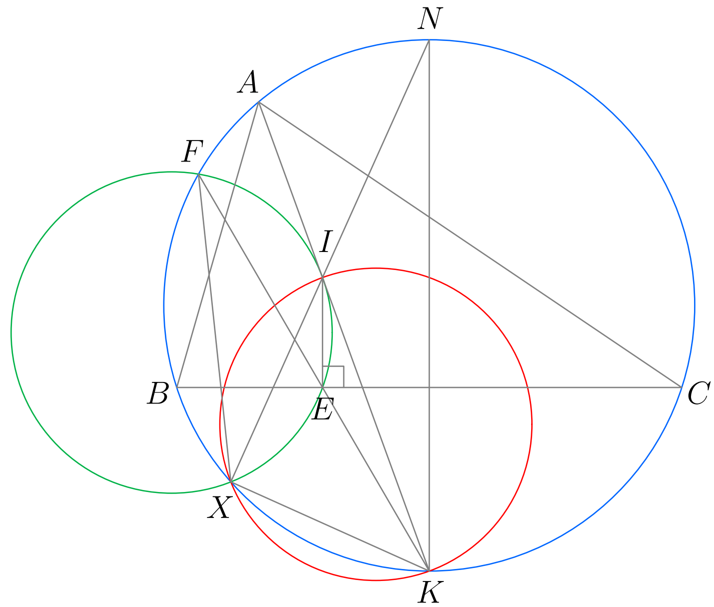
$$N(8,3) = N(7,2) + 1 + N(7,3) = 63 + 1 + 98\,;$$

following through the discussion above, Heidi would begin by turning over the 64th card. If it is yellow, then she wins; if it is red, then she considers the preceding $N(7,2) = 63$ cards and acts in a similar way; if it is green, then she considers the subsequent $N(7,3) = 98$ cards and acts similarly. As an example, suppose that the yellow card is in position 53: then the following would be the cards turned over by Heidi, and their colour:

$$64\,,\ 22\,,\ 38\,, 49\,,\ 56\,,\ 52\,,\ 54\,,\ 53\,.$$

Q1774 Let $\triangle ABC$ be a triangle in which $|AB| \neq |AC|$, let I be its in-centre, and let E be the foot of the perpendicular from I to side BC. The line through A and I intersects the circumcircle of $\triangle ABC$ again at a point K. The line KE intersects the circumcircle again at point F. Prove that the following three circles have a common point: the circumcircle of $\triangle ABC$, the circle with KI as its diameter, and the circumcircle of $\triangle FIE$.

SOLUTION Since $AB \neq AC$, the in-centre does not lie on a diameter of the circumcircle of $\triangle ABC$; therefore, this circumcircle and the circle with KI as its diameter have two points of intersection, one of which is K; let X be the other. We shall show that X is also on the circumcircle of $\triangle FIE$, so that the three circles have a common point, as required.



Let N be the point of intersection, other than X, of the line XI with the circumcircle of $\triangle ABC$. Now $\angle KXN$ equals $\angle KXI$, which is a right angle (angle in the semicircle on IK), so KN is a diameter of the circumcircle of $\triangle ABC$. Moreover, the line KA bisects the angle at A, so the point K bisects the arc BC; so the diameter KN is perpendicular to the chord BC, and therefore parallel to EI. This gives

$$\angle XIE = \angle XNK = \angle XFK = \angle XFE \,.$$

Thus, the points X, F, I, E lie on a circle, and our proof is complete.

Q1775 Show that for any positive integer n, and for any $2n$ real numbers

$$a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$$

lying between -1 and 1 (inclusive), we have

$$|a_1a_2\cdots a_n - b_1b_2\cdots b_n| \leq |a_1 - b_1| + |a_2 - b_2| + \cdots + |a_n - b_n| \,.$$

SOLUTION The inequality is obviously true when $n = 1$; we proceed by mathematical induction. So, suppose that n is an integer such that

$$|a_1a_2\cdots a_n - b_1b_2\cdots b_n| \leq |a_1 - b_1| + |a_2 - b_2| + \cdots + |a_n - b_n|$$

for all choices of $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ between -1 and 1; our aim is to show that a similar inequality holds for any choice of real numbers $a_1, a_2, \ldots, a_n, a_{n+1}, b_1, b_2, \ldots, b_n, b_{n+1}$ between -1 and 1. Let

$$c = \frac{a_{n+1} + b_{n+1}}{2} \quad \text{and} \quad d = \frac{a_{n+1} - b_{n+1}}{2}\,;$$

then we have $a_{n+1} = c + d$ and $b_{n+1} = c - d$, and so

$$\begin{aligned} &|a_1a_2\cdots a_na_{n+1} - b_1b_2\cdots b_nb_{n+1}| \\ &\qquad = |a_1a_2\cdots a_n(c+d) - b_1b_2\cdots b_n(c-d)| \\ &\qquad = |c(a_1a_2\cdots a_n - b_1b_2\cdots b_n) + (a_1a_2\cdots a_n + b_1b_2\cdots b_n)d| \\ &\qquad \le |c|\,|a_1a_2\cdots a_n - b_1b_2\cdots b_n| + |a_1a_2\cdots a_n + b_1b_2\cdots b_n|\,|d|\,. \end{aligned}$$

Since all the a_k and b_k are between -1 and 1, we have

$$|a_{n+1} + b_{n+1}| \le 2\,, \quad |a_1a_2\cdots a_n| \le 1\,, \quad |b_1b_2\cdots b_n| \le 1$$

and so

$$|c| \le 1\,, \quad |a_1a_2\cdots a_n + b_1b_2\cdots b_n| \le 2\,.$$

Remembering also that $d = (a_{n+1} - b_{n+1})/2$ and using the inductive hypothesis, we have from above

$$\begin{aligned} &|a_1a_2\cdots a_na_{n+1} - b_1b_2\cdots b_nb_{n+1}| \\ &\qquad \le |a_1a_2\cdots a_n - b_1b_2\cdots b_n| + 2\left|\frac{a_{n+1} - b_{n+1}}{2}\right| \\ &\qquad \le |a_1 - b_1| + |a_2 - b_2| + \cdots + |a_{n+1} - b_{n+1}|\,, \end{aligned}$$

as required.

Q1776 A virus is located at a point of an infinite two–dimensional grid of 1×1 squares. One can heal a virus at a point if it has at least four non–infected adjacent points – "adjacent" means adjacent either horizontally, vertically or diagonally. However, healing a virus at one point leads to infecting four adjacent points. Is it possible to reach a situation in which there are no viruses at distance 3 or less from the location of the original virus?

SOLUTION Let $(0,0)$ be the original point. We put the number $w^{|i|+|j|}$ at the point (i,j), where w is a number between 0 and 1 which we shall specify later, and call this the "weight" of point (i,j). The pattern of weights will look like this,

$$\begin{array}{ccccccc} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & w^4 & w^3 & w^2 & w^3 & w^4 & \cdots \\ \cdots & w^3 & w^2 & w & w^2 & w^3 & \cdots \\ \cdots & w^2 & w & \color{red}{1} & w & w^2 & \cdots \\ \cdots & w^3 & w^2 & w & w^2 & w^3 & \cdots \\ \cdots & w^4 & w^3 & w^2 & w^3 & w^4 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

where the red entry is the location of the initial virus.

Suppose that a point at (i,j) has weight x. The weights of adjacent points depend on whether (i,j) is the origin, in which case $x = 1$; or directly left, right, above or below the origin (on one of the "axes"); or elsewhere.

(i,j) at the origin		
w^2	w	w^2
w	1	w
w^2	w	w^2

(i,j) "on an axis"		
xw^2	xw	xw^2
xw	x	xw
x	xw^{-1}	x

(i,j) elsewhere		
x	xw	xw^2
xw^{-1}	x	xw
xw^{-2}	xw^{-1}	x

Consider what happens when a virus at a point of weight x on an axis is healed. The total weight of all infected points is decreased by x, but also increased by a minimum of w^2x and wx, twice each. (Remember that w is less than 1, so $w < w^{-1}$.) The total increase in weight, therefore, is at least $(2w^2 + 2w - 1)x$; and choosing the value

$$w = \frac{-1+\sqrt{3}}{2}\,,$$

the root of the quadratic, means that the total increase cannot be negative in this case: that is, the total weight can only remain the same or decrease. If a virus at an "elsewhere" point of weight x is healed, then at least one of the newly infected points must have weight x or more, so once again, the total weight of infected points cannot decrease. If a virus at the origin is healed, then the resulting total weight is at least $4w^2$. Therefore, the total weight of infected points can never fall below $4w^2$. We shall show, however, that the total weight of all points at distance greater than 3 from the origin is less than $4w^2$; since the total weight of infected points is more than this, there must always be at least one infected point at distance 3 or less from the origin. (And so the answer to the question asked is "no".)

So, how can we add up the weights of the required points? To begin, consider the 5×5 diagram at the beginning of our solution. Starting to multiply out a product, we have

$$\begin{aligned}&\Big(w^2+w+1+w+w^2\Big)\Big(w^2+w+1+w+w^2\Big)\\ &\qquad = \Big(w^4+w^3+w^2+w^3+w^4\Big) + \Big(w^3+w^2+w+w^2+w^3\Big) + \cdots\,,\end{aligned}$$

and when we have finished, the right–hand side will include exactly the numbers in the square. Their sum is therefore

$$S_2 = (w^2+w+1+w+w^2)^2\,.$$

In the same way, the sum of all weights in a $(2n+1)\times(2n+1)$ square centred at $(0,0)$ is

$$S_n = (w^n+\cdots+w+1+w+\cdots+w^n)^2 = \Big(1+2w\,\frac{1-w^n}{1-w}\Big)^2\,.$$

We obtained the expression on the right–hand side by summing a geometric series – we leave this to readers as an exercise. The sum of all weights in the whole plane is the value that this approaches when n approaches infinity,

$$S_\infty = \Big(1 + \frac{2w}{1-w}\Big)^2 = \Big(\frac{1+w}{1-w}\Big)^2 .$$

The points at distance 3 or less from $(0,0)$ are the points in the 5×5 square just considered, and the four points of weight w^3 at $(0,3)$ and so on; therefore the total weight of all other points is

$$S_\infty - S_2 - 4w^3 = \Big(\frac{1+w}{1-w}\Big)^2 - \big(1 + 2w(1+w)\big)^2 - 4w^3 .$$

To complete the proof, we need to show that this is less than $4w^2$, as foreshadowed above. Perhaps the best way to do this is to subtract $4w^2$ and multiply by $(1-w)^2$, so that we need to prove

$$(1+w)^2 - (1+2w+2w^2)^2(1-w)^2 - 4w^2(1+w)(1-w)^2 < 0 .$$

This looks nasty, but remember that we know that $2w^2 + 2w - 1 = 0$. So $1 + 2w + 2w^2$ is simply 2; and we can partially multiply out the last term: $2w(1+w) = 2w + 2w^2 = 1$. So the left–hand side is

$$\begin{aligned}
(1+w)^2 - 4(1-w)^2 - 2w(1-w)^2 &= -3 + 8w + w^2 - 2w^3 \\
&= (2w^2 + 2w - 1)\Big(-w + \frac{3}{2}\Big) + 4w - \frac{3}{2} \\
&= \frac{8w-3}{2} \\
&= \frac{-7 + 4\sqrt{3}}{2} .
\end{aligned}$$

This is indeed negative, and we are finished.

Q1777 Let n be an integer, $n \geq 5$. Prove that if the distinct prime factors of n are $p_1, p_2, \ldots, p_k$, then the product

$$A = n^2\Big(1 - \frac{1}{p_1^2}\Big)\Big(1 - \frac{1}{p_2^2}\Big) \cdots \Big(1 - \frac{1}{p_k^2}\Big)$$

is equal to 24 times an integer.

SOLUTION By writing down all the different prime factors of n once each and then collecting any repetitions, we have

$$n = p_1 p_2 \cdots p_k m$$

for some integer m. Then we have

$$A = m^2(p_1^2 - 1)(p_2^2 - 1) \cdots (p_k^2 - 1) ; \qquad (*)$$

this is clearly an integer, and we have to show that it is a multiple of 24. Now let p be a prime. We have

$$\begin{aligned} p=2 \quad &\Rightarrow \quad p^2-1=3\,; \\ p=3 \quad &\Rightarrow \quad p^2-1=8\,; \\ p>3 \quad &\Rightarrow \quad p=6q\pm 1 \quad \text{for some integer } q \\ &\Rightarrow \quad p^2-1=12q(3q\pm 1)\,. \end{aligned}$$

In the last case either q is even and so p^2-1 is a multiple of 24; or q is odd, so $3q\pm 1$ is even, and p^2-1 is still a multiple of 24. It follows that if n has any prime factor $p_j\geq 5$, then the factor p_j^2-1 is a multiple of 24, so is A, and we are finished. If n has both 2 and 3 as factors, then one of the factors in $(*)$ is 3, another is 8, and the whole is once again a multiple of 24. The only remaining cases are when the prime factors of n are 2 only, or 3 only. In the first case we have $n=2^a$ for some a; and $a\geq 3$ because $n\geq 5$; so

$$A=2^{2a}\Bigl(1-\frac{1}{4}\Bigr)=3\times 2^{2a-2}$$

which is a multiple of 24. Finally we consider $n=3^b$ with $b\geq 2$; and

$$A=3^{2b}\Bigl(1-\frac{1}{9}\Bigr)=8\times 3^{2b-2}\,,$$

a multiple of 24.

Q1778 Jack is playing cards with three other players seated around a table. A pack of 52 cards is being used. Jack deals cards to the person on his left, then to the next person around the table, and so on. But instead of dealing single cards, he gives 5 to the first person, 2 to the next, then $1,1,1,2,1$; and then, continuing around the table, repeats the sequence $5,2,1,1,1,1,2,1$ as many times as needed until the whole pack has been dealt out. Jack is amazed to find that despite this irregular deal, each of the four players ends up with the same number of cards: a fact that you can easily confirm.

Next day, Jack is playing again and decides on another unconventional way of dealing. He has in mind a sequence of non–negative integers $a_1,a_2,\ldots,a_n$ which add up to 13. He deals cards in batches of $a_1,a_2,\ldots,a_n$ to successive players; and then continues around the table with the same sequence repeatedly until all 52 cards have been dealt. Prove that if n is odd, then every player ends up with the same number of cards; while if n is even, then this is not so.

SOLUTION We refer to Jack's initial distribution of $a_1,a_2,\ldots,a_n$ cards as the first phase of the deal; then the same again as the second phase, and so on. Let's call the person on Jack's left Queenie. She receives the first a_1 cards in the first phase. The first a_1 cards in the second phase go to the person n places around the table from Queenie. If n is odd, then this person cannot be Queenie herself, nor the person opposite her; so they must be one of the people next to her, in one or the other direction around the circle. The first a_1 cards in the third and fourth phases go to the following people in the same direction around the table. That is, every one of the four players receives a_1 cards at some stage of the deal. But for exactly the same reasons, each player also receives a_2 additional cards, and so on. Therefore, the total number of cards received by each player is $a_1 + a_2 + \cdots + a_n = 13$.

On the other hand, suppose that n is even. Then $2n$ is a multiple of 4. Therefore, the numbers of cards Queenie receives in the first and second phases are exactly duplicated in the third and fourth phases; and so the total number of cards she receives is even. In particular, Queenie (and likewise each of the other three players) cannot have received 13 cards, and so not all have received the same number.

Q1779 Simplify the sum

$$S_n = 1^2 \binom{n+1}{1}^2 + 2^2 \binom{n+1}{2}^2 + \cdots + (n+1)^2 \binom{n+1}{n+1}^2 .$$

SOLUTION We give two solutions for this problem. First, we shall prove the result by combinatorial arguments; second, by using a generating function approach. We begin by noting that, for any n and k satisfying $0 \le k \le n$, we have

$$(k+1)\binom{n+1}{k+1} = (n+1)\binom{n}{k} . \qquad (*)$$

To see this, imagine a sports club with $n+1$ members; a team consisting of $k+1$ players is to be chosen, and one member of the team is to be designated captain. If we choose the whole team first, and then one of the team members as captain, then the number of possible outcomes is the left–hand side of $(*)$. On the other hand, if we choose one member of the club as captain and then the remaining k team players from the remaining n club members, then the number of possible outcomes is the right–hand side of $(*)$. As we have just solved the same problem in two ways, the answers must be the same, and the equality $(*)$ is proved. We can therefore write

$$\begin{aligned} S_n &= \sum_{k=0}^{n} (k+1)^2 \binom{n+1}{k+1}^2 \\ &= \sum_{k=0}^{n} (n+1)^2 \binom{n}{k}^2 \\ &= (n+1)^2 \sum_{k=0}^{n} \binom{n}{k}^2 . \end{aligned}$$

Now consider another sports club, consisting of n women and n men; we wish to select a team of n players. One way to do this is, for any k from 0 to n, to choose k of the n women and omit k of the n men. So the number of choices is

$$\sum_{k=0}^{n} \binom{n}{k}^2 .$$

But another way is to simply pick n players out of all $2n$ members; the number of options is $\binom{2n}{n}$, and this must be equal to the above sum. Therefore, we get

$$S_n = (n+1)^2 \binom{2n}{n} .$$

Solutions received from Pranav Kudumula and Henry Ricardo, both essentially using the above method.

Alternative solution. We can also prove this result by writing down the binomial expansion for $(1+x)^{n+1}$, differentiating and then squaring:

$$\begin{aligned}
(1+x)^{n+1} &= \sum_{m=0}^{n+1} \binom{n+1}{m} x^m \\
&\Rightarrow \quad (n+1)(1+x)^n = \sum_{m=0}^{n+1} m \binom{n+1}{m} x^{m-1} \\
&\Rightarrow \quad (n+1)^2(1+x)^{2n} = \left(\sum_{m=0}^{n+1} m \binom{n+1}{m} x^{m-1} \right)^2 .
\end{aligned}$$

The coefficient of x^n on the left–hand side is $(n+1)^2\binom{2n}{n}$. To obtain terms in x^n on the right–hand side, we multiply an x^{k-1} term by an x^{n+1-k} term, where $k = 1, 2, \ldots, n+1$. These terms are obtained by taking $m = k$ and $m = n - k + 2$, respectively, in the previous equation; and so the coefficient is

$$\sum_{k=1}^{n+1} k \binom{n+1}{k} (n-k+2) \binom{n+1}{n-k+2} .$$

However, using the factorial expression for binomial coefficients gives

$$(n-k+2)\binom{n+1}{n-k+2} = \frac{(n+1)!}{(n-k+1)!\,(k-1)!} = k \binom{n+1}{k} ,$$

and so the coefficient is

$$\sum_{k=1}^{n+1} k^2 \binom{n+1}{k}^2 ,$$

which is the number S_n. Since we have shown that the coefficient of x^n in the expansion of $(n+1)^2(1+x)^{2n}$ is S_n, and also that it is $(n+1)^2\binom{2n}{n}$, these two expressions are equal, and we are finished.

Q1780 Consider the cubic polynomial

$$f(x) = x^3 + \alpha x^2 + (\alpha - \beta)x - 1 ,$$

where α and β are real numbers.

(a) Let $\beta = 2$. Find a value of α for which $f(x)$ has one rational root and two irrational roots.

(b) Find a value of β with the following property: for every real α, either $f(x)$ has three rational roots, or $f(x)$ has three real irrational roots.

SOLUTION For (a), it is not hard to find that $f(-1) = 0$, regardless of the value of α. So we have the factorisation

$$f(x) = (x + 1)\big(x^2 + (\alpha - 1)x - 1\big)$$

and the roots

$$x_1 = -1 , \quad x_2, x_3 = \frac{-(\alpha - 1) \pm \sqrt{(\alpha - 1)^2 + 4}}{2} .$$

Because of the square root, we would generally expect the second expression to be irrational, so we'll just guess a value for α (and guess again if we're unlucky). If $\alpha = 0$, then the roots are

$$x_1 = -1 , \quad x_2, x_3 = \frac{1 \pm \sqrt{5}}{2} ;$$

these are one rational, two irrational, as required.

For the general case, the required property may be expressed as follows: for every α, if one root of $f(x)$ is rational, then every root of $f(x)$ is rational. So, suppose that x_1 is a rational root of $f(x)$. Then we have

$$x_1^3 + \alpha x_1^2 + (\alpha - \beta)x_1 - 1 = 0$$

and therefore

$$(x_1^2 + x_1)\alpha = 1 + \beta x_1 - x_1^3 . \qquad (*)$$

If $x_1 = 0$, then this says $0 = 1$ which is impossible; if $x_1 = -1$, then we have $\beta = 2$, and we already know that this is not a suitable value. So we may assume that $x_1 \neq 0$ and $x_1 \neq -1$. Let x_2, x_3 be the other roots of $f(x)$; then Vieta's formulas give

$$x_1 + x_2 + x_3 = -\alpha \quad \text{and} \quad x_1x_2x_3 = 1 . \qquad (**)$$

Now, x_2 and x_3 are solutions to the equation $(t - x_2)(t - x_3) = 0$. We expand this; use $(**)$ to eliminate x_2 and x_3; multiply by $x_1(x_1 + 1)$; and use $(*)$ to eliminate α. Doing the algebra carefully, this results in the quadratic equation

$$x_1(x_1 + 1)t^2 + (x_1^2 + \beta x_1 + 1)t + (x_1 + 1) = 0 .$$

In order to ensure rational solutions, we would like the discriminant

$$(x_1^2 + \beta x_1 + 1)^2 - 4x_1(x_1 + 1)^2 = x_1^4 + \cdots + 1$$

to be the square of a polynomial in x_1. This polynomial must have the form $x_1^2 + \gamma x_1 \pm 1$ for some constant γ. If we expand

$$(x_1^2 + \gamma x_1 + 1)^2 = (x_1^2 + \beta x_1 + 1)^2 - 4x_1(x_1 + 1)^2$$

and equate coefficients, then we find

$$2\gamma = 2\beta - 4\,, \quad \gamma^2 + 2 = \beta^2 - 6$$

with solution $\beta = 3$, $\gamma = 1$. If we do the same with $x_1^2 + \gamma x_1 - 1$, then we obtain $\beta = 2$, which does not work, as we already know. So we choose $\beta = 3$ and solve the quadratic in t to give

$$\begin{aligned} t &= \frac{-(x_1^2 + 3x_1 + 1) \pm \sqrt{(x_1^2 + 3x_1 + 1)^2 - 4x_1(x_1 + 1)^2}}{2x_1(x_1 + 1)} \\ &= \frac{-(x_1^2 + 3x_1 + 1) \pm (x_1^2 + x_1 + 1)}{2x_1(x_1 + 1)}\,. \end{aligned}$$

Therefore, the other roots of $f(x)$ are

$$x_2 = -\frac{1}{x_1 + 1}\,, \quad x_3 = -\frac{x_1 + 1}{x_1}\,,$$

which are rational. To complete the problem, we should check that when $\beta = 3$, the cubic polynomial has three real roots. In this case, we have $f(-1) = 1$ and $f(0) = -1$; the conclusion follows from sketching the graph of $y = f(x)$.