

Trigonometric ratios can prove the Pythagorean Theorem

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1 Pythagorean Theorem

The Pythagorean Theorem occupies a central role in geometry and analysis. Consider triangle ABC right-angled at C , with side lengths $|AB| = c$, $|BC| = a$ and $|CA| = b$. Then

$$c^2 = a^2 + b^2, \quad (1)$$

is well known as the Pythagorean Theorem [1, 2, 3, 4, 6, 7, 8, 9, 10, 11]. Dividing both sides of Eq. (1) by c^2 yields $1 = \frac{a^2}{c^2} + \frac{b^2}{c^2}$. Setting $\sin \theta = \frac{b}{c}$ and $\cos \theta = \frac{a}{c}$ for $\angle ABC = \theta$, it follows that Eq. (1) is equivalent to the Pythagorean Identity $1 = \cos^2 \theta + \sin^2 \theta$.

Historically, because the trigonometric ratios $\sin \theta$ and $\cos \theta$ are defined by side-length ratios in a right triangle, any proof of the Pythagorean Theorem based on these ratios has been deemed circular. This perspective motivates our search for a genuinely non-circular approach. For this problem, Jackson and Johnson [4] bypass this obstacle by combining a geometric construction with an infinite geometric series and the Double-Angle Identity, marking a breakthrough in the field [3, 4, 8, 10]. Jackson and Johnson innovatively combined geometry and analysis in their approach. Their key insight was that combining an infinite geometric series with the Double-Angle Identity circumvents circular reasoning. Nonetheless, earlier works have provided proofs based exclusively on infinite geometric series [2, 6, 9]. This raises the question: is the novelty of Jackson and Johnson's approach attributable primarily to trigonometric identities or to their use of infinite series?

To gain deeper insight into their method, which resolved a problem unsolved for nearly two millennia, we analyse whether their success derives from the infinite series, the Double-Angle Identity, or a combination thereof. Although prior research has established proofs of the Pythagorean Theorem via infinite series [2, 6, 7, 9], relatively few have focused exclusively on the Double-Angle Identity to prove the theorem. Specifically, by isolating the Double-Angle Identity, we aim to demonstrate that infinite series are not essential for non-circular trigonometric proofs of the Pythagorean Theorem.

Hence, we investigate whether the Pythagorean Theorem can be derived solely from the Double-Angle Identity, without invoking infinite series. Furthermore, we attempt to prove the Pythagorean Theorem by employing the Angle-Bisector Theorem

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in conjunction with the double-angle approach. In addition, we discuss the connection between these two proofs, derive a new relational formula for trigonometric ratios, and use it to prove the Pythagorean Theorem. We also compare our approaches with classical Euclidean proofs to highlight their novelty.

This paper is organized as follows. Section 2 presents the first proof via an isosceles triangle construction and the Tangent Double-Angle Identity. Section 3 offers an alternative proof using the Angle-Bisector Theorem. In Section 4, we derive the Double-Angle Identity itself from the Angle-Bisector Theorem, and Section 5 develops a new relation to obtain a third proof. In the final section, we conclude our work and devote future works.

2 First Proof

We begin by constructing an isosceles triangle that avoids reliance on series expansions. Figure 1 illustrates the key angles and lengths in this construction, and provides intuitive geometric insight into the double-angle relationship. Let ABD be an isosceles triangle with $|AB| = |AD|$ and $\angle BAD < \frac{\pi}{2}$. Dropping the perpendicular from B to AD at C yields a right triangle ABC with $|AB| = c$, $|BC| = a$ and $|CA| = b$; hence, $|CD| = |AD| - |AC| = c - b$. Setting $\angle BAD = 2\theta$ gives $\angle ADB = \angle ABD = \frac{\pi}{2} - \theta$ and $\angle DBC = \frac{\pi}{2} - \angle BDC = \frac{\pi}{2} - \angle ADB = \theta$. Then $\tan 2\theta = \frac{|BC|}{|CA|} = \frac{a}{b}$ and $\tan \theta = \frac{|CD|}{|BC|} = \frac{c-b}{a}$. Substituting into the Tangent Double-Angle Identity $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$ yields

$$\frac{a}{b} = \frac{2 \left(\frac{c-b}{a} \right)}{1 - \left(\frac{c-b}{a} \right)^2}, \quad (2)$$

and rearranging Eq. (2) by cross-multiplication immediately yields $c^2 = a^2 + b^2$.

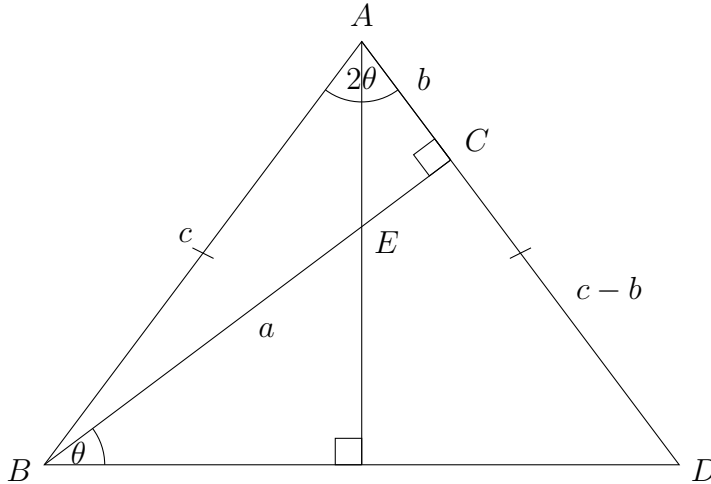


Figure 1: Isosceles-triangle construction.

This construction offers an intuitive geometric interpretation and a non-circular proof of the Pythagorean Theorem that relies solely on the Tangent Double-Angle Identity, without invoking infinite series. In particular, it demonstrates that, between the two prior ideas – (1) the infinite series and (2) the Double-Angle Identity – the latter is the essential ingredient.

Finally, two points should be noted here. First, since the above proof also covers the case $a = b$ (i.e. when triangle ABC is an isosceles right triangle), it constitutes an extension of Jackson and Johnson’s prior work. Next, it turns also out that this proof does not make use of the Pythagorean Identity $1 = \cos^2 \theta + \sin^2 \theta$.

3 Second Proof

To validate the generality of our approach, we employ classical theorems in a novel manner. In the previous section, we proved the Pythagorean Theorem using the Tangent Double-Angle Identity. However, since we have not discussed whether that identity itself is derived from the Pythagorean Identity $1 = \cos^2 \theta + \sin^2 \theta$, we cannot yet assert that this proof is entirely free of circularity. Hence, it is necessary to re-examine the proof by employing a different approach. Here, we reuse Figure 1 but invoke the Angle-Bisector Theorem to furnish a second proof. Notably, our method reveals a deeper connection between angle-bisector properties and trigonometric relations.

Since we set $\angle BAD = 2\theta$ in the first proof, let E be the intersection of the bisector of $\angle BAD$ with side BC . This similarity reveals a deeper structural link between angle bisectors and trigonometric ratios. Then right triangles EAC and DBC are similar, since $\angle EAC = \angle DBC = \theta$. By the Angle-Bisector Theorem, $|AC| : |AB| = |EC| : |EB|$ and $|EC| + |EB| = |BC|$, and we compute $|EC| = |BC| \frac{|AC|}{|AC| + |AB|} = \frac{ab}{b+c}$. It follows that $\tan \theta = \frac{|EC|}{|AC|} = \frac{\frac{ab}{b+c}}{b} = \frac{a}{b+c}$ and $\tan \theta = \frac{|DC|}{|BC|} = \frac{c-b}{a}$. Thus, equating the two expressions for $\tan \theta$ immediately gives $c^2 = a^2 + b^2$.

In Section 4, we explain how the Tangent Double-Angle Identity follows from the Angle-Bisector Theorem and basic geometry, without ever invoking the Pythagorean Identity $1 = \cos^2 \theta + \sin^2 \theta$. Moreover, a comparison of both proofs highlights the versatility of elementary trigonometric techniques. We now turn our attention to the implications of the unified identity.

4 Derivation of the Double-Angle Identity

Having established two independent proofs, we now derive standard identities from first principles. In this section, we here derive the Tangent Double-Angle Identity $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$ from the Angle-Bisector Theorem, without invoking the Pythagorean Identity $1 = \cos^2 \theta + \sin^2 \theta$. As in Figure 2, consider right triangle ABC with $\angle ACB = \frac{\pi}{2}$. We set $|BC| = 1$ and let D be the intersection of the bisector of $\angle CBA = 2\theta$ with side AC . From right triangle BCD , one finds $|CD| = \tan \theta$. We also set $|AD| = x$

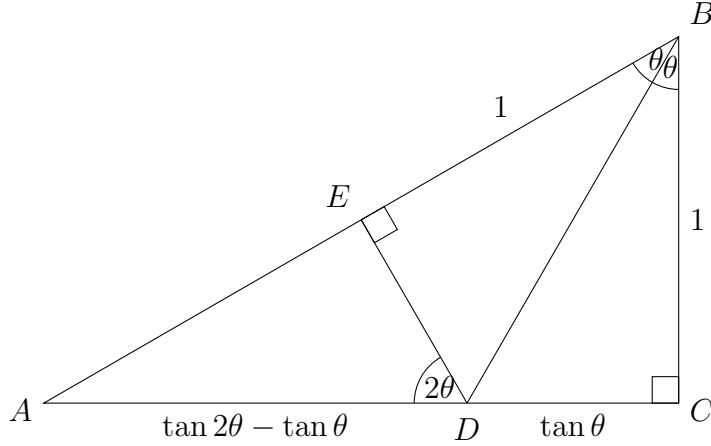


Figure 2: Angle-Bisector Theorem and Double-Angle Identity.

and $|AB| = y$. By the Angle-Bisector Theorem, $|AB| : |BC| = |AD| : |CD|$; hence, $x = y \tan \theta$. Since $|BC| = 1$, rewriting $\tan 2\theta$ in terms of side-lengths yields that

$$\tan 2\theta = \frac{|AC|}{|BC|} = |AD| + |CD| = x + \tan \theta = (y + 1) \tan \theta. \quad (3)$$

To complete the derivation, we use congruence arguments as follows. Next, drop the perpendicular from D to side AB and denote its foot by E . Since $\angle DBC = \angle DBE = \theta$, the right triangles BCD and BED are congruent by the hypotenuse angle condition; hence, $|BC| = |BE| = 1$ and $|CD| = |ED| = \tan \theta$. Therefore, the right triangles AED and ACB are similar, so $\angle ADE = 2\theta$ and

$$\tan 2\theta = \frac{|AE|}{|ED|} = \frac{|AB| - |BE|}{|ED|} = \frac{y - 1}{\tan \theta}. \quad (4)$$

Eliminating y between Eq. (3) and Eq. (4) yields the standard Double-Angle Identity $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$.

Importantly, our derivation does not depend on the Pythagorean Identity⁴. Note that, although we use the definition of tangent, e.g. $\tan 2\theta = \frac{AC}{BC} = \frac{|AE|}{|ED|}$, we do not rely on the Pythagorean Identity $1 = \cos^2 \theta + \sin^2 \theta$. Thus, one can derive the relation between $\tan 2\theta$ and $\tan \theta$ without circularity (see Appendix A), so it is erroneous to claim that all trigonometric proofs of the Pythagorean Theorem are circular by nature; a careful, case-by-case analysis yields deeper insight into each proof.

For completeness, we also derive the double-angle identities for sine and cosine by constructing an appropriate right triangle and applying the law of sines and cosines. As in Figure 3, we prepare a right triangle ABC with $\angle ACB = \frac{\pi}{2}$ and $\angle BAC = \theta (< \frac{\pi}{4})$. Next, place a point D on side BC so that $\angle ABD = \theta$, and for simplicity assume that

⁴A similar figure is shown in [7] but is not used as a proof based on the Double-Angle Identity.

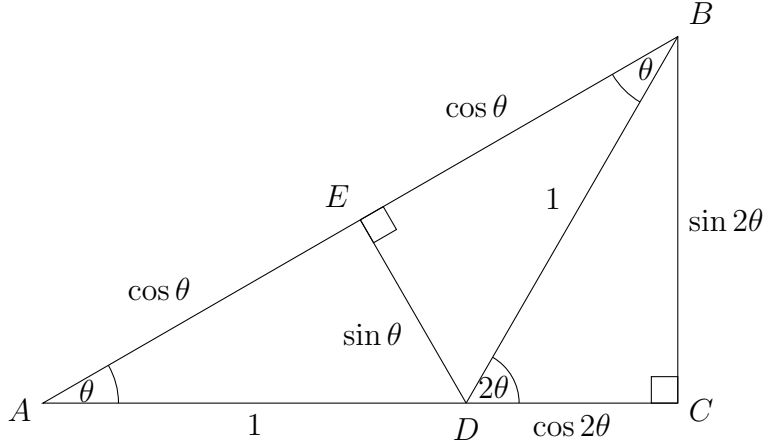


Figure 3: The Double-Angle Identity.

$|AD| = 1$. Triangle ABD is isosceles with $\angle ABD = \angle BAD = \theta$ and $|AB| = |AD|$. Since $\angle BDC = 2\theta$, we obtain $|BC| = \sin 2\theta$ and $|CD| = \cos 2\theta$. Let E be the foot of the perpendicular from D to side AB . Triangle ADE is a right triangle with $\angle DEA = \frac{\pi}{2}$, $\angle DAE = \theta$ and $|AD| = 1$. Hence, $|AE| = \cos \theta$ and $|AB| = 2|AE| = 2\cos \theta$, so the sine of angle $\angle BAC = \theta$ is $\sin \theta = \frac{|BC|}{|AB|} = \frac{\sin 2\theta}{2\cos \theta}$ which yields the Double-Angle Identity for sine, $\sin 2\theta = 2\sin \theta \cos \theta$. Similarly, $\cos \theta = \frac{|AC|}{|AB|} = \frac{|AD| + |CD|}{|AB|} = \frac{1 + \cos 2\theta}{2\cos \theta}$, giving the Double-Angle Identity for cosine, $\cos 2\theta = 2\cos^2 \theta - 1$.

5 Third Proof

We extend our analysis by exploring alternative trigonometric relations. In this section, we employ the right triangle ABC from Figure 2 to derive additional trigonometric identities and present a novel proof of the Pythagorean Theorem. One recalls that $|BC| = 1$ and $|CD| = |ED| = \tan \theta$. Since $\angle ABC = 2\theta$ and $|BC| = 1$, we have $\tan 2\theta = \frac{AC}{BC}$; hence, $|AC| = \tan 2\theta$. Similarly, $\cos 2\theta = \frac{|BC|}{|AB|}$ implies that $|AB| = \frac{1}{\cos 2\theta}$. Next, the area S of triangle ABD can be expressed as $S = \frac{1}{2}|BC||AD| = \frac{1}{2}|AB||ED|$. Since $|AD| = |AC| - |CD| = \tan 2\theta - \tan \theta$, one obtains $\frac{\tan \theta}{\cos 2\theta} = \tan 2\theta - \tan \theta$ and derives $\tan \theta = \frac{\sin 2\theta}{1 + \cos 2\theta}$.

Moreover, we apply this identity to the configurations introduced in Sections 2 and 3. In the right triangle ABC with $\angle ACB = \frac{\pi}{2}$, $\angle BAC = 2\theta$ and side lengths $|AB| = c$, $|BC| = a$ and $|CA| = b$, we have $\sin 2\theta = \frac{|BC|}{|AB|} = \frac{a}{c}$ and $\cos 2\theta = \frac{|CA|}{|AB|} = \frac{b}{c}$. In the right triangle BDC with $\angle BCD = \frac{\pi}{2}$, $\angle DBC = \theta$, $|BC| = a$ and $|CD| = c - b$, one takes $\tan \theta = \frac{|CD|}{|BC|} = \frac{c-b}{a}$. Substituting into $\tan \theta = \frac{\sin 2\theta}{1 + \cos 2\theta}$ immediately yields $c^2 = a^2 + b^2$. Thus, this section demonstrates that, beyond the standard Double-Angle Identity $\tan 2\theta = \frac{2\tan \theta}{1 - \tan^2 \theta}$, the Pythagorean Theorem can also be proved using alternative trigonometric relations.

6 Conclusion

In this work, we have highlighted the primacy of double-angle identities. In trigonometric proofs of the Pythagorean Theorem, we have focused on two main approaches: (1) the infinite geometric series argument and (2) double-angle identities, following Jackson and Johnson's work [4]. Our results demonstrate that the essential ingredient is the tangent double-angle identities, rather than infinite series constructions. Specifically, we prove the Pythagorean Theorem by employing the Double-Angle Identity for tangent, $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$, and we show that the Angle-Bisector Theorem yields an equivalent derivation. Moreover, we derive the trigonometric relation $\tan\theta = \frac{\sin 2\theta}{1+\cos 2\theta}$ and use it to prove the Pythagorean Theorem.

Although this study and prior works have provided proofs based on the double-angle identities, applying these double-angle identities in future works to novel geometric configurations may yield additional non-circular proofs of the Pythagorean Theorem. Furthermore, to develop new trigonometric proofs of the Pythagorean Theorem, it will be helpful to integrate triple-angle and half-angle formulas, as well as sum-to-product and product-to-sum identities, in a manner that avoids circular reasoning. Further exploration of half-angle formulas may yield additional insights.

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A Appendix

For completeness, we derive the sine and cosine addition formulas geometrically. To do so, we first derive the Law of Cosine, in Subsection A.1 of this appendix. In Subsection A.2, we derive the addition angles formula for sine and cosine. Then, in Subsection A.3, we prove the Pythagorean Identity $1 = \cos^2 \theta + \sin^2 \theta$ directly using a cyclic quadrilateral and the angle addition formulas [11]. Finally, Subsection A.4 presents an exercises for the reader, to consolidate their understanding of the trigonometric techniques discussed here.

A.1 The Law of Cosine and another trigonometric identity

The Law of Cosine is shown in Figure 4. In this figure, $|TC||CS| = |CQ||CA|$; that is, $(c - a)(c + a) = (2c \cos A - b)b$, from which we derive the Law of Cosine, $c^2 = a^2 + b^2 - 2bc \cos A$.

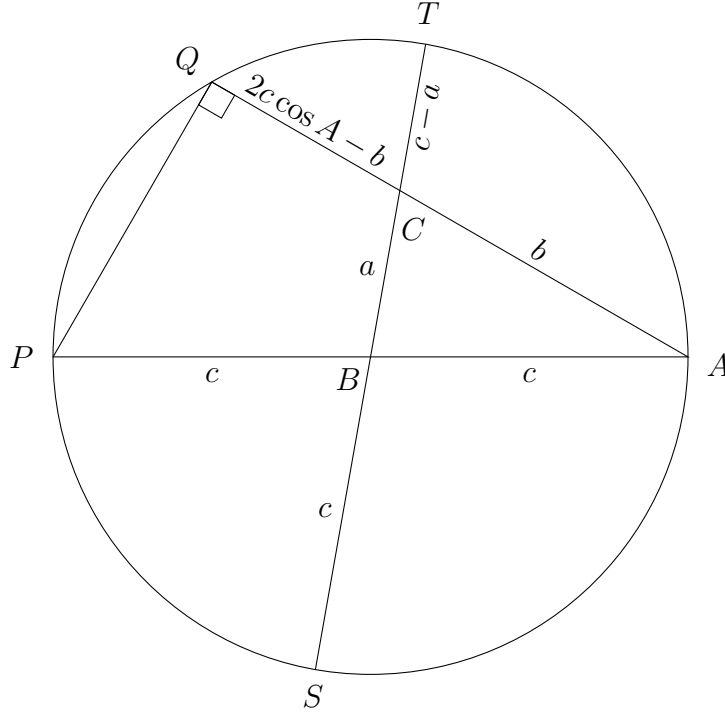


Figure 4: The Law of Cosine and the Intersecting Chords Theorem [5].

We now prove the trigonometric identity

$$\frac{1}{\cos^2 \alpha} = 1 + \tan^2 \alpha \quad (5)$$

without invoking the Pythagorean Identity $1 = \cos^2 \alpha + \sin^2 \alpha$. Consider the right triangle ABD with $\angle BAD = \frac{\pi}{2}$ in Figure 5, and let C be the foot of the perpendicular

from A to BD . Let $\angle ABC = \alpha$ and $|BC| = 1$; then $|AC| = \tan \alpha$ and $|AB| = \frac{1}{\cos \alpha}$. Since $\angle DAC = \alpha$ and $\angle DCA = \frac{\pi}{2}$, triangles ABC and DAC are similar, so the identities $|CD| = |AC| \tan \alpha = \tan^2 \alpha$ and $|BD| = |BC| + |CD| = 1 + \tan^2 \alpha$ are derived. Thus, we have $\cos \alpha = \frac{|AB|}{|BD|} = \frac{\frac{1}{\cos \alpha}}{1 + \tan^2 \alpha}$, whence Eq. (5) is derived.

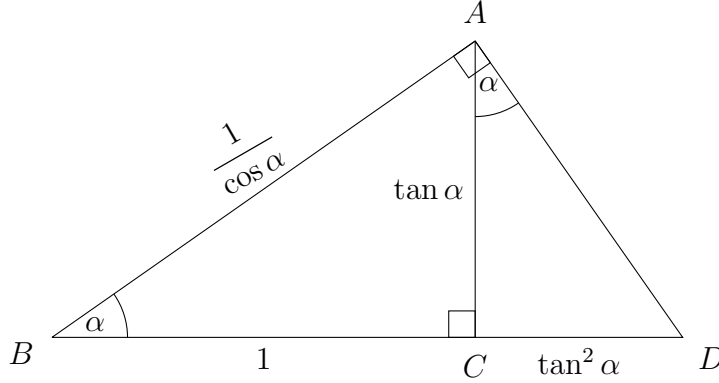


Figure 5: $\frac{1}{\cos^2 \alpha} = 1 + \tan^2 \alpha$.

A.2 Angle addition and subtraction formulas

Consider Figure 6, in which triangle ABD satisfies $\angle BAD < \frac{\pi}{2}$ and $\angle BDA < \frac{\pi}{2}$, and let C be the foot of the perpendicular from B to side AD .

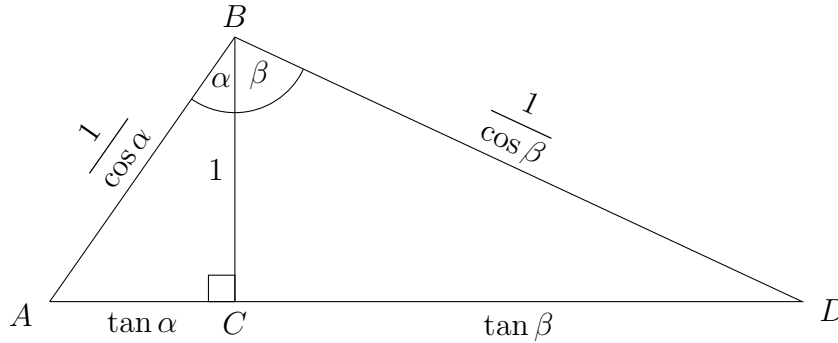


Figure 6: Geometric proof of the Addition Angles Identity.

To simplify the discussion, let $|BC| = 1$, $\angle ABC = \alpha$ and $\angle DBC = \beta$, respectively. Then $|AC| = \tan \alpha$, $|AB| = \frac{1}{\cos \alpha}$, $|CD| = \tan \beta$ and $|BD| = \frac{1}{\cos \beta}$. Applying the Law of Sines to triangle ABD yields $\frac{|AD|}{\sin(\alpha + \beta)} = \frac{|AB|}{\sin(\frac{\pi}{2} - \beta)} = \frac{|BD|}{\sin(\frac{\pi}{2} - \alpha)}$. A straightforward rearrangement using the identities $\angle ABD = \angle ABC + \angle DBC = \alpha + \beta$,

$|AD| = |AC| + |CD| = \tan \alpha + \tan \beta$ and $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos \alpha$ then gives the Angle Addition Identity for Sine,

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta.$$

Next, applying the Law of Cosines and Eq. (5) to triangle ABD at angle $\angle ABD = \alpha + \beta$ yields the Angle Addition Identity for Cosine:

$$\begin{aligned} \cos(\alpha + \beta) &= \frac{|AB|^2 + |BD|^2 - |AD|^2}{2|AB||BD|} \\ &= \frac{\cos \alpha \cos \beta}{2} \left(\frac{1}{\cos^2 \alpha} + \frac{1}{\cos^2 \beta} - (\tan \alpha + \tan \beta)^2 \right) \\ &= \cos \alpha \cos \beta - \sin \alpha \sin \beta. \end{aligned} \tag{6}$$

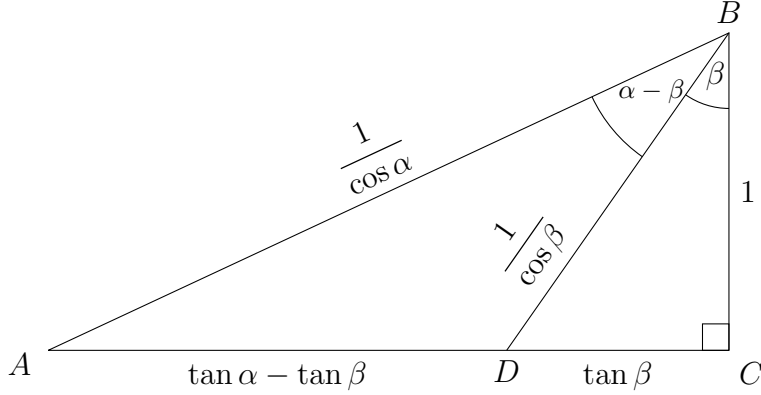


Figure 7: Geometric proof of the Subtraction Angles Identity.

Furthermore, we derive the angle subtraction identities. As shown in Figure 7, consider the right triangle ABC with $\angle ACB = \frac{\pi}{2}$ and $\angle ABC = \alpha$, and let $|BC| = 1$. Moreover, let D be the point on AC so that $\angle DBC = \beta (< \alpha)$. Then $|AB| = \frac{1}{\cos \alpha}$, $|BD| = \frac{1}{\cos \beta}$, $|AC| = \tan \alpha$ and $|CD| = \tan \beta$. The Law of Sines applied to triangle ABD yields $\frac{|AD|}{\sin(\alpha - \beta)} = \frac{|BD|}{\sin(\frac{\pi}{2} - \alpha)} = \frac{|AB|}{\sin(\frac{\pi}{2} + \beta)}$. By the identities $\angle ABD = \angle ABC - \angle DBC = \alpha - \beta$ and $|AD| = |AC| - |CD| = \tan \alpha - \tan \beta$, we find the Angle Subtraction Identity for Sine,

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta.$$

Finally, applying the Law of Cosines and Eq. (5) to triangle ABD at $\angle ABD = \alpha - \beta$ gives the Angle Subtraction Identity for Cosine:

$$\cos(\alpha - \beta) = \frac{|AB|^2 + |BD|^2 - |AD|^2}{2|AB||BD|}$$

$$\begin{aligned}
&= \frac{\cos \alpha \cos \beta}{2} \left(\frac{1}{\cos^2 \alpha} + \frac{1}{\cos^2 \beta} - (\tan \alpha - \tan \beta)^2 \right) \\
&= \cos \alpha \cos \beta + \sin \alpha \sin \beta.
\end{aligned} \tag{7}$$

Thus, the angle addition and subtraction formulas for sine and cosine can be derived without relying on the Pythagorean Identity $1 = \cos^2 \theta + \sin^2 \theta$.

A.3 A proof of the Pythagorean Identity

We now prove the Pythagorean Identity directly using a cyclic quadrilateral and the angle addition formulas [11].

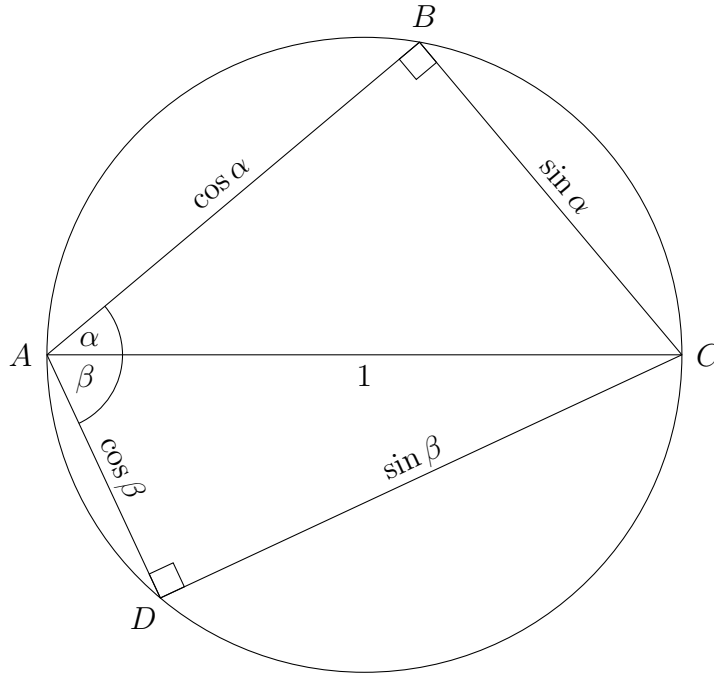


Figure 8: Geometric derivation of the Pythagorean Identity $1 = \cos^2 \theta + \sin^2 \theta$.

Figure 8 shows a quadrilateral $ABCD$ inscribed in a circle of radius $\frac{1}{2}$ and diameter $|AC| = 1$, so that $\angle ABC = \angle ADC = \frac{\pi}{2}$. We set $\angle BAC = \alpha$ and $\angle DAC = \beta$. In right triangles ABC and ADC , one has $|AB| = \cos \alpha$, $|BC| = \sin \alpha$, $|AD| = \cos \beta$ and $|DC| = \sin \beta$. Then, the area S of quadrilateral $ABCD$ can be written in two ways: $S = \frac{1}{2}|AB||BC| + \frac{1}{2}|AD||DC|$ and $S = \frac{1}{2}|AB||AD| \sin(\alpha + \beta) + \frac{1}{2}|BC||DC| \sin(\pi - \alpha - \beta)$, by the Angle Addition Identity for Sine and the identities $\angle BAD = \angle BAC + \angle DAC = \alpha + \beta$, $\angle BCD = \pi - \angle BAD = \pi - \alpha - \beta$ and $\sin(\pi - \theta) = \sin \theta$. Thus, one finds

$$0 = \sin \alpha \cos \alpha (\cos^2 \beta + \sin^2 \beta - 1) + \sin \beta \cos \beta (\cos^2 \alpha + \sin^2 \alpha - 1). \tag{8}$$

Since Eq. (8) holds for any α, β , it turns out that $1 = \cos^2 \theta + \sin^2 \theta$.

A.4 Exercise

The following exercise consolidates the trigonometric techniques discussed above. Figure 9 shows the right triangle ABC with $\angle ABD = \angle BAD = \angle DBF = \theta$ ($< \frac{\pi}{6}$), $\angle BDF = 2\theta$, $\angle BFC = 3\theta$, $\angle ACB = \angle AED = \frac{\pi}{2}$, $|AD| = |BD| = 1$, $|AB| = 2 \cos \theta$, $|BC| = \sin 2\theta$ and $|CD| = \cos 2\theta$.

- Find the lengths $|BF|$ and $|DF|$.
- Show that $\sin 3\theta + \sin \theta = 2 \cos \theta \sin 2\theta$.
- Using Figure 1, and noting that $\sin 2\theta = \frac{a}{c}$, $\cos 2\theta = \frac{b}{c}$ and $\tan \theta = \frac{c-b}{a}$, prove the Pythagorean Theorem.

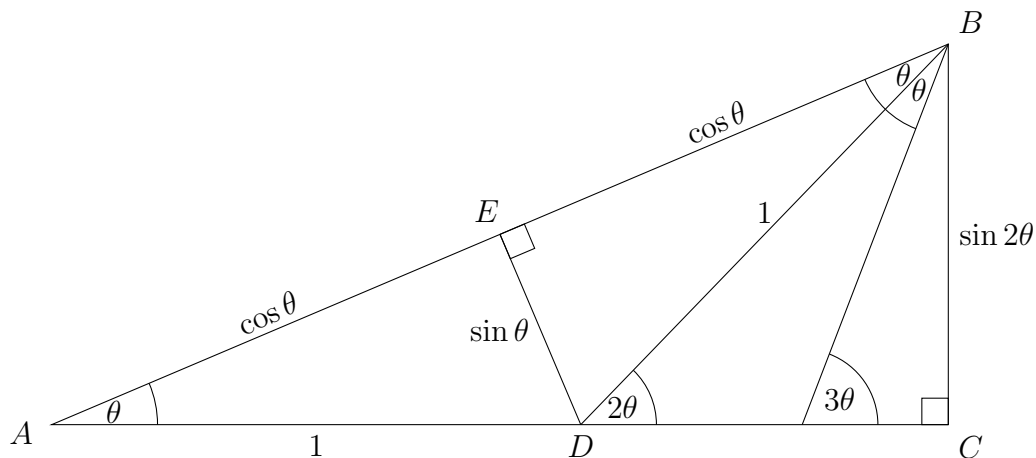


Figure 9: Exercise.