

A novel method of evaluating the Riemann zeta function at even natural numbers using elementary calculus

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1 Introduction

The Basel Problem involves finding the convergence value of the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

and is one of the most famous classical problems of the 17th century. Leonhard Euler had first revealed that the series converges to $\frac{1}{6}\pi^2$, a groundbreaking discovery that further established a connection between the Riemann zeta function and trigonometric functions.

Many mathematicians have since uncovered various properties of the Riemann zeta function ζ , defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (1)$$

for $s \in \mathbb{C}$ such that $\text{Re}(s) > 1$, but it remains a mystery in numerous aspects, including the Riemann Hypothesis. Although not many values of the Riemann zeta function are known to this day, the values at even natural numbers can be deduced using the relation with the Bernoulli numbers B_n , which was first proposed by Euler [6, 1] and is given by

$$\zeta(2k) = (-1)^{k+1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k}. \quad (2)$$

One known recursive form is represented by

$$\zeta(2n) = \frac{2}{2n+1} \sum_{k=1}^{n-1} \zeta(2k) \zeta(2n-2k) \quad (3)$$

which lets us find values of $\zeta(2n)$ in terms of the values of ζ at smaller even numbers. This relation has been proved by using the infinite series expansion of $f(x) = \pi x \cot \pi x$ and then comparing the coefficients of the corresponding powers of x after differentiating f with respect to x , as shown by Koecher [7]. Also, Williams [10] provides an

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alternative proof by expressing each product $\zeta(2n)\zeta(2n - 2k)$ as a double sum truncated at N , applying the geometric series formula to calculate the inner summation, and then letting $N \rightarrow \infty$ once the series has been suitably modified.

Moreover, using (2), we observe that (3) is equivalent to

$$B_{2n} = -\frac{1}{2n+1} \sum_{k=1}^{n-1} \binom{2n}{2k} B_{2k} B_{2n-2k}.$$

Several proofs for this recursive formula exist, such as in [3].

This article proposes a different approach to the Basel problem and extends this process to derive the recursive formula (3) using only elementary calculus techniques. This novel method utilises the sequence of functions F_n which we introduce below.

Definition 1. For $n \in \mathbb{N}$, let F_n be a sequence of functions on $\mathbb{R}$ defined by

$$F_n(x) = \sum_{k=1}^{\infty} \frac{1}{(x^2 + k^2)^n}.$$

We start by examining its properties and convergence to ensure mathematical rigour and then proceed to derive its recursive form. Finding values of $F_n(0)$ not only yields the solution to the Basel problem when $n = 1$ but also provides a recursive form of the Riemann zeta function at even natural numbers with a straightforward proof.

2 Some properties of series of functions

In this section, we provide some lemmas concerning the convergence of general series of functions before we apply them specifically to F_n . Throughout this article, we will often write $\sum f_n$ to denote the infinite series

$$\sum_{n=1}^{\infty} f_n = f_1 + f_2 + f_3 + \cdots$$

associated to the sequence of functions $\{f_n\}$. We may similarly write $\sum a_n$ to denote the infinite series associated to the sequence of real numbers $\{a_n\}$.

We state the following lemmas without proof. Interested readers may refer to [8, Theorem 7.10] for Lemma 2, [2, Theorem 9.7] for Lemma 3, [9, Theorem 4.4.20] for Lemma 4 and [8, Corollary 7.16] for Lemma 5.

Lemma 2 (Weierstrass M -test). Suppose $\{f_n\}$ is a sequence of functions defined on $E \subseteq \mathbb{R}$ and

$$|f_n(x)| \leq M_n$$

for all $x \in E$ and $n \in \mathbb{N}$. Then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

Lemma 3. If each f_n is continuous on E and $\sum f_n$ converges uniformly to f on E , then f is also continuous on E .

Lemma 4. Suppose that each f_n is continuously differentiable on $[a, b]$, $\sum f_n(x_0)$ converges for some x_0 in $[a, b]$, and $\sum f'_n$ converges uniformly on $[a, b]$. Then $\sum f_n$ converges uniformly to a differentiable function f and moreover, for all $x \in (a, b)$,

$$f'(x) = \sum_{n=1}^{\infty} f'_n(x).$$

Lemma 5. If $\{f_n\}$ is a sequence of Riemann-integrable functions on $[a, b]$ and the infinite series

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

converges uniformly on $[a, b]$, then

$$\int_a^b f(x) dx = \sum_{n=1}^{\infty} \int_a^b f_n(x) dx.$$

The above lemmas are useful to study the continuity and differentiability of F_n .

Theorem 6. For each $n \in \mathbb{N}$, F_n is continuous and differentiable on $\mathbb{R}$.

Proof. Consider the sequence of functions $\{f_k\}$ on $\mathbb{R}$ for $k \in \mathbb{N}$, where

$$f_k(x) = \frac{1}{(x^2 + k^2)^n}.$$

Observe that for all $x \in \mathbb{R}$ and for $k \in \mathbb{N}$,

$$|f_k(x)| = \left| \frac{1}{(x^2 + k^2)^n} \right| \leq \frac{1}{k^{2n}}.$$

The series

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \cdots$$

converges since it is a p -series with $p = 2n > 1$. By Lemma 2, the series $\sum f_k$ converges uniformly for all $x \in \mathbb{R}$. Also, $\{f_k\}$ is a sequence of functions which are continuous for all $x \in \mathbb{R}$. Therefore, $F_n = \sum f_k$ is continuous on $\mathbb{R}$ by Lemma 3.

Now, consider the sequence $\{f'_k\}$ where

$$f'_k(x) = \frac{-2nx}{(x^2 + k^2)^{n+1}}.$$

We have that, for all $x \in \mathbb{R}$ and $k \in \mathbb{N}$,

$$|f'_k(x)| = \frac{2n|x|}{(x^2 + k^2)^{n+1}} \leq \frac{n(x^2 + k^2)}{k(x^2 + k^2)^{n+1}} = \frac{n}{k(x^2 + k^2)^n} \leq \frac{n}{k^{2n+1}}$$

where we used the fact that $2k|x| \leq x^2 + k^2$ in the first inequality above. Similar to the previous argument, we may deduce that the series $\sum f'_k$ converges uniformly on $\mathbb{R}$ by Lemma 2.

Since the series $\sum f_k(0)$ converges, by Lemma 4 we must have that F_n is differentiable on $\mathbb{R}$ and $F'_n(x) = \sum f'_k(x)$ holds for all $x \in \mathbb{R}$. $\square$

3 Solution to the Basel problem using F_1

Before giving our proof to the Basel problem, we first present the following lemma without proof. We refer readers to the proof by Eberlein [5] or Ciaurri [4] for the sine product formula.

Lemma 7. *For all $z \in \mathbb{C}$, we have that*

$$\sinh z = z \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{k^2 \pi^2} \right).$$

Theorem 8 (The Basel Problem). *The sum of the reciprocals of all perfect squares is*

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Proof. By the definition of F_n , the Basel problem is equivalent to finding the value of $F_1(0)$. We first note that

$$2xF_1(x) = \sum_{k=1}^{\infty} \frac{2x}{x^2 + k^2}. \quad (4)$$

Now we focus on the behaviour of F_1 in the neighbourhood of 0 by restricting the domain of F_1 to $D = (-R, R) \setminus \{0\}$ for some $R > 0$. We then have

$$\left| \frac{2x}{x^2 + k^2} \right| < \frac{2R}{k^2}$$

and the convergence in (4) is uniform by Lemma 2. Thus on D ,

$$\begin{aligned} \int 2xF_1(x) dx &= \sum_{k=1}^{\infty} \int \frac{2x}{x^2 + k^2} dx \\ &= \sum_{k=1}^{\infty} \left[\ln \left(1 + \frac{x^2}{k^2} \right) \right] + C \\ &= \ln \left(\prod_{k=1}^{\infty} \left(1 + \frac{x^2}{k^2} \right) \right) + C \\ &= \ln \left(\frac{\sinh(\pi x)}{\pi x} \right) + C \end{aligned} \quad (5)$$

where the interchange of the summation and the integral in the first equation is justified by Lemma 5 and the last equation is due to Lemma 7.

If we differentiate both sides of (5) with respect to x , then we obtain

$$2xF_1(x) = \frac{\pi x \coth(\pi x) - 1}{x}$$

which gives us an alternative formula for F_1 on D as

$$F_1(x) = \frac{\pi x \coth(\pi x) - 1}{2x^2}. \quad (6)$$

Using the continuity of F_1 established in Theorem 6, the original Basel Problem reduces to finding

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = F_1(0) = \lim_{x \rightarrow 0} F_1(x) = \lim_{x \rightarrow 0} \frac{\pi x \coth(\pi x) - 1}{2x^2}$$

and this limit exists and can be found by several applications of l'Hôpital's Rule. More precisely,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\pi x \coth(\pi x) - 1}{2x^2} &= \lim_{x \rightarrow 0} \frac{\pi x \cosh(\pi x) - \sinh(\pi x)}{2x^2 \sinh(\pi x)} \\ &\stackrel{\text{L}}{=} \lim_{x \rightarrow 0} \frac{\pi^2 x \sinh(\pi x)}{4x \sinh(\pi x) + 2x^2 \pi \cosh(\pi x)} \\ &= \lim_{x \rightarrow 0} \frac{\pi^2 \sinh(\pi x)}{4 \sinh(\pi x) + 2x \pi \cosh(\pi x)} \\ &\stackrel{\text{L}}{=} \lim_{x \rightarrow 0} \frac{\pi^3 \cosh(\pi x)}{4\pi \cosh(\pi x) + 2\pi \cosh(\pi x) + 2x\pi^2 \sinh(\pi x)} \\ &= \frac{1}{6} \pi^2 \end{aligned}$$

which gives the solution to the Basel Problem. Here the notation $\stackrel{\text{L}}{=}$ indicates the use of l'Hôpital's Rule. $\square$

4 Deriving the recursive form involving $\zeta(2n)$

The values $\zeta(2n)$ of the Riemann zeta function (1) for even natural numbers can be thought of as

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = F_n(0).$$

Similar to the previous section, getting a handle of F_n may allow us to find $\zeta(2n)$.

To that end, we first note that

$$\frac{dF_n(x)}{dx} = \frac{d}{dx} \left(\sum_{k=1}^{\infty} \frac{1}{(x^2 + k^2)^n} \right) = \sum_{k=1}^{\infty} \frac{-2nx}{(x^2 + k^2)^{n+1}} = -2nx F_{n+1}(x)$$

where the derivative and summation can be swapped by the uniform convergence established in Theorem 6. Hence for $x \neq 0$, the relation

$$F_{n+1}(x) = -\frac{F'_n(x)}{2nx} \quad (7)$$

holds and allows us to find a recurrence relation for F_n . We show how one may obtain F_2 from F_1 in the following theorem.

Theorem 9. *In some neighbourhood of 0 except possibly at $x = 0$, we have that*

$$F_2(x) = F_1(x)^2 + \frac{3F_1(x) - \frac{1}{2}\pi^2}{2x^2}.$$

Proof. From (6), we have that

$$\pi \coth \pi x = 2xF_1(x) + \frac{1}{x}. \quad (8)$$

Differentiating both sides of (8) and applying (7), we have

$$-\pi^2 \operatorname{csch}^2 \pi x = 2F_1(x) + 2xF_1'(x) - \frac{1}{x^2} = 2F_1(x) - 4x^2 F_2(x) - \frac{1}{x^2}.$$

We now use the identity $\coth^2 \pi x - \operatorname{csch}^2 \pi x = 1$ and (8) to get

$$\pi^2 - \left(2xF_1(x) + \frac{1}{x}\right)^2 = 2F_1(x) - 4x^2 F_2(x) - \frac{1}{x^2}$$

and finally, by making $F_2(x)$ the subject,

$$F_2(x) = F_1(x)^2 + \frac{3F_1(x) - \frac{1}{2}\pi^2}{2x^2}$$

as claimed. $\square$

One may then find a recurrence relation for F_n , $n \geq 3$ by applying the same technique. We provide the next three recurrence relations for F_3 , F_4 and F_5 below:

$$\begin{aligned} F_3(x) &= F_1(x)F_2(x) + \frac{5F_2(x) - 2F_1(x)^2}{4x^2}; \\ F_4(x) &= \frac{1}{3}F_2(x)^2 + \frac{2}{3}F_1(x)F_3(x) + \frac{7F_3(x) - 4F_1(x)F_2(x)}{6x^2}; \\ F_5(x) &= \frac{1}{2}F_1(x)F_4(x) + \frac{1}{2}F_2(x)F_3(x) + \frac{9F_4(x) - 2F_2(x)^2 - 4F_1(x)F_3(x)}{8x^2}. \end{aligned} \quad (9)$$

The following theorem captures a general recurrence relation for F_n when $n \geq 3$.

Theorem 10. *For $n \geq 3$, we have that*

$$F_n(x) = \frac{1}{n-1} \sum_{k=1}^{n-1} F_k(x)F_{n-k}(x) + \frac{(2n-1)F_{n-1}(x) - 2A_{n-1}(x)}{2(n-1)x^2} \quad (10)$$

where

$$A_n(x) = \sum_{k=1}^{n-1} F_k(x)F_{n-k}(x),$$

valid in some neighbourhood around $x = 0$ except possibly at $x = 0$.

Proof. We shall prove the statement by mathematical induction. The base case $n = 3$ gives

$$F_3(x) = F_1(x)F_2(x) + \frac{5F_2(x) - 2F_1(x)^2}{4x^2}$$

which holds by (9). Take any $n \geq 3$ and assume that (10) holds. Differentiating both sides of (10) with respect to x yields

$$\begin{aligned} F'_n(x) &= \frac{1}{n-1} \left(\sum_{k=1}^{n-1} F'_k(x)F_{n-k}(x) + \sum_{k=1}^{n-1} F_k(x)F'_{n-k}(x) \right) \\ &\quad + \frac{(2n-1)F'_{n-1}(x) - 2A'_{n-1}(x)}{2(n-1)x^2} - \frac{(2n-1)F_{n-1}(x) - 2A_{n-1}(x)}{(n-1)x^3}. \end{aligned}$$

By applying (7) and using the induction assumption, we may obtain

$$\begin{aligned} F_{n+1}(x) &= \frac{1}{n(n-1)} \left(\sum_{k=1}^{n-1} kF_{k+1}(x)F_{n-k}(x) + \sum_{k=1}^{n-1} (n-k)F_k(x)F_{n-k+1}(x) \right) \\ &\quad + \frac{(n-1)(2n-1)F_n(x) - 2B(x)}{2n(n-1)x^2} + \frac{(n-1)F_n(x) - A_n(x)}{n(n-1)x^2}, \end{aligned}$$

where

$$B(x) = \sum_{k=1}^{n-2} kF_{k+1}(x)F_{n-k-1}(x) + \sum_{k=1}^{n-2} (n-k-1)F_k(x)F_{n-k}(x).$$

By some elementary but tedious algebraic manipulation, we finally obtain

$$F_{n+1}(x) = \frac{1}{n} \sum_{k=1}^n F_k(x)F_{n+1-k}(x) + \frac{(2n+1)F_n(x) - 2A_n(x)}{2nx^2}$$

which finishes the induction. We conclude that the recurrence relation (10) holds for all $n \geq 3$. $\square$

Theorem 11 (The recursive form of $\zeta(2n)$). *For $n \geq 2$, the following recurrence relation holds:*

$$\zeta(2n) = \frac{2}{2n+1} \sum_{k=1}^{n-1} \zeta(2k)\zeta(2n-2k).$$

Proof. From (10), we have that

$$\lim_{x \rightarrow 0} \frac{(2n-1)F_{n-1}(x) - 2A_{n-1}(x)}{2(n-1)x^2} = \lim_{x \rightarrow 0} F_n(x) - \frac{1}{n-1} \lim_{x \rightarrow 0} \sum_{k=1}^{n-1} F_k(x)F_{n-k}(x). \quad (11)$$

Using the continuity of F_n at $x = 0$, the right-hand side of (11) exists so it follows that

$$(2n-1)F_{n-1}(x) - 2A_{n-1}(x) = (2n-1)F_{n-1}(x) - 2 \sum_{k=1}^{n-2} F_k(x)F_{n-k-1}(x) \rightarrow 0$$

as $x \rightarrow 0$ since the denominator $2(n-1)x^2 \rightarrow 0$ as $x \rightarrow 0$.

Again by the continuity of F_n , if $n \geq 3$ we get that

$$\begin{aligned} 0 &= \lim_{x \rightarrow 0} \left((2n-1)F_{n-1}(x) - 2 \sum_{k=1}^{n-2} F_k(x)F_{n-k-1}(x) \right) \\ &= (2n-1)F_{n-1}(0) - 2 \sum_{k=1}^{n-2} F_k(0)F_{n-k-1}(0). \end{aligned} \quad (12)$$

Shifting the index n in the summation in (12) above gives

$$\zeta(2n) = F_n(0) = \frac{2}{2n+1} \sum_{k=1}^{n-1} F_k(0)F_{n-k}(0) = \frac{2}{2n+1} \sum_{k=1}^{n-1} \zeta(2k)\zeta(2n-2k)$$

for $n \geq 2$, proving the claim. $\square$

Using Theorem 11, we list down below the values of $\zeta(2n) = F_n(0)$ for $1 \leq n \leq 5$:

$$\begin{aligned} \zeta(2) &= \frac{\pi^2}{6}; \\ \zeta(4) &= \frac{2}{5}F_1(0)^2 = \frac{\pi^4}{90}; \\ \zeta(6) &= \frac{4}{7}F_1(0)F_2(0) = \frac{\pi^6}{945}; \\ \zeta(8) &= \frac{2}{9}(2F_1(0)F_3(0) + F_2(0)^2) = \frac{\pi^8}{9450}; \\ \zeta(10) &= \frac{2}{11}(2F_1(0)F_4(0) + 2F_2(0)F_3(0)) = \frac{\pi^{10}}{93555}. \end{aligned}$$

5 Conclusion

Through this process, we have derived a recurrence relation of the Riemann zeta function at even natural numbers by first establishing a recurrence for F_n . While other methods exist for proving this formula, we emphasise that our approach introduces a new technique based on elementary calculus. We expect that the function F_n may have potential extension to the complex domain and offer alternative approaches beyond traditional methods.

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