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Complex numbers are (un)real?

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1 Introduction

One of the most famous and celebrated inequalities for positive real numbers is the *AM-GM-HM Inequality* which states that

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}} \tag{1}$$

for all positive real numbers $a_1, \ldots, a_n$, with equality if and only if $a_1 = \cdots = a_n$.

In this article, we provide a surprising spin-off of the AM-GM-HM Inequality and other inequalities for complex numbers with a fixed modulus. Our results are motivated by a geometric observation. Although elementary in nature and possibly already known, the results presented here were established independently. A result of the same flavour, but derived under different assumptions and using more sophisticated tools, can be found in [1].

2 Convex functions and Jensen's Inequality

We first define what convex and concave functions are and then state Jensen's Inequality, which relates to convex functions and will be used in the later part of the article.

Definition 1 (Convex and concave functions)**.** *Let* $I \subseteq \mathbb{R}$.
A function $f\colon I \to \mathbb{R}$ *is* convex *on* I *if, for any real numbers* $x, y \in I$ *and* $\alpha \in [0, 1]$,

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)\,.$$

A function $f\colon I \to \mathbb{R}$ *is* concave *on* I *if, for any real numbers* $x, y \in I$ *and* $\alpha \in [0, 1]$,

$$f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha) f(y)\,.$$

We now state a result regarding convex and concave functions, namely Jensen's Inequality. We omit the proof; interested readers can consult [2].

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Theorem 2 (Jensen's Inequality)**.** *Let $I \subseteq \mathbb{R}$, let $h\colon I \to \mathbb{R}$ be a function and let $w_1, \dots, w_n$ be n positive real numbers. For all real numbers $x_1, \dots, x_n \in I$,*

$$h\Big(\frac{w_1x_1 + \cdots + w_nx_n}{w_1 + \cdots + w_n}\Big) \leq \frac{w_1h(x_1) + \cdots + w_nh(x_n)}{w_1 + \cdots + w_n}$$

if h is convex and

$$h\Big(\frac{w_1x_1 + \cdots + w_nx_n}{w_1 + \cdots + w_n}\Big) \geq \frac{w_1h(x_1) + \cdots + w_nh(x_n)}{w_1 + \cdots + w_n}$$

if h is concave. Equality holds if and only if $x_1 = \cdots = x_n$ or h is linear.

We shall give a proof of the AM-GM-HM Inequality using Jensen's Inequality.

Theorem 3. *Let $a_1, \dots, a_n$ be positive real numbers. We have that*

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}}\,,$$

and equality holds if and only if $a_1 = \ldots = a_n$.

Proof. Using Jensen's Inequality and the fact that $\ln$ is concave on $(0, \infty)$, we have

$$\ln\Big(\frac{a_1 + \cdots + a_n}{n}\Big) \geq \frac{\ln a_1 + \cdots + \ln a_n}{n} = \frac{\ln a_1 \cdots a_n}{n}\,.$$

By exponentiating both sides and using the fact that $x \mapsto e^x$ is an increasing map on $\mathbb{R}$, we obtain the AM-GM Inequality. The GM-HM Inequality follows by applying the AM-GM Inequality to the positive real numbers $\frac{1}{a_1}, \dots, \frac{1}{a_n}$. $\square$

The above inequality can be generalised to the *Power Mean Inequality*, expressed in the following theorem. A full proof can be found in [2]. The proof again utilises Jensen's Inequality and the fact that $\ln$ is concave on $(0, \infty)$.

Theorem 4 (Power Mean Inequality)**.** *Let $a_1, \dots, a_n$ be positive real numbers. For any $k \in \mathbb{R}$, define*

$$P_k = P_k(a_1, \dots, a_n) := \begin{cases} \big(\frac{1}{n}(a_1^k + \cdots + a_n^k)\big)^{\frac{1}{k}} & \text{if } k \neq 0, \\ (a_1 \cdots a_n)^{\frac{1}{n}} & \text{if } k = 0. \end{cases}$$

Then $k \mapsto P_k$ is a non-decreasing function; that is, $P_s \leq P_r$ whenever $r \leq s$.

As a remark, the Power Mean Inequality naturally admits a weighted generalisation wherein one associates a positive weight w_i to each a_i such that $w_1 + \cdots + w_n = 1$.

3 Results

We now state our results. We first state and prove our AM-GM-HM Inequality variant for two complex numbers and then generalise it for n complex numbers.

Theorem 5. *Let $a, b \in \mathbb{C}$ with $|a| = |b| > 0$. Then*

$$\left|\frac{a+b}{2}\right| \leq |\sqrt{ab}| \leq \left|\frac{2ab}{a+b}\right| . \tag{2}$$

Proof. We provide two proofs, one geometric and one algebraic. We give an illustration for the geometric proof in Figure 1 below.

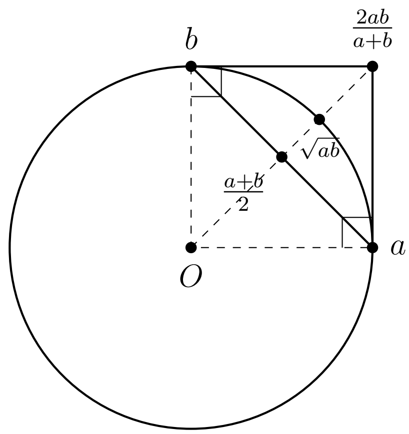

Figure 1: The AM-GM-HM Inequality corresponding to the points a and $b = ia$

Consider the circle in the Argand plane centred at origin with radius $\rho = |a| = |b|$. Note that $\left|\frac{1}{2}(a+b)\right|$ is the distance from the origin of the midpoint of the chord joining the points a and b. Clearly, this midpoint lies inside the circle unless $a = b$, in which case the midpoint is on the circle. Using properties of the modulus function, we have $|\sqrt{ab}| = \rho$.

Now, $\left|\frac{2ab}{a+b}\right|$ represents the distance from the origin to the point of intersection of tangents drawn to the circle at a and b. This point of intersection lies outside the circle unless $a = b$, in which case the intersection is again on the circle. This finishes our geometric proof.

Alternatively, we may use the Triangle Inequality. For the left-hand inequality of (2), we have that

$$\left|\frac{a+b}{2}\right| = \frac{1}{2}|a+b| \leq \frac{1}{2}\big(|a| + |b|\big) = \frac{1}{2}(\rho + \rho) = \rho = \left|\sqrt{ab}\right| .$$

For the right-hand inequality, we have that

$$\left|\frac{2ab}{a+b}\right| \geq \frac{2\rho^2}{2\rho} = \rho = \left|\sqrt{ab}\right|$$

since $|a+b| \leq 2\rho$, and this completes the proof. $\square$

One might be tempted to think that this contradicts the AM-GM-HM Inequality (1) but it does not. Note that the AM-GM-HM Inequality in (1) demands that the numbers be positive. Since our inequality holds for complex numbers with the same modulus, the only case possible for both to occur is $a = b = \rho$. In this case, both inequalities in (2) are in fact equalities.

Indeed, it is possible to extend the result above for n complex numbers with the same modulus.

Theorem 6. *Let $z_1, \ldots, z_n$ be n complex numbers with the same modulus $\rho > 0$. Then*

$$\left|\frac{z_1 + \cdots + z_n}{n}\right| \leq \left|(z_1 \cdots z_n)^{\frac{1}{n}}\right| \leq \left|\frac{n}{\frac{1}{z_1} + \cdots + \frac{1}{z_n}}\right|. \tag{3}$$

Proof. Note that $\left|(z_1 z_2 \cdots z_n)^{\frac{1}{n}}\right| = \rho$. For the left-hand side inequality of (3), we have that

$$\left|\frac{z_1 + \cdots + z_n}{n}\right| \leq \frac{1}{n}\sum_{i=1}^{n} |z_i| = \rho\,.$$

For the right-hand side inequality of (3), we have to show that

$$\rho \leq \left|\frac{n}{\frac{1}{z_1} + \cdots + \frac{1}{z_n}}\right|.$$

This again follows from the triangle inequality since $\left|\frac{1}{z_i}\right| = \frac{1}{\rho}$ for all $i = 1, \ldots, n$. □

An argument similar to the one above shows that this is not a contradiction to the AM-GM-HM Inequality.

We shall now provide a generalisation of the Power Mean Inequality.

Theorem 7 (Power Mean Inequality for complex numbers)**.** *Let $z_1, \ldots, z_n$ be n complex numbers with the same modulus $\rho > 0$. For any $k \in \mathbb{R}$, define*

$$P_k = P_k(z_1, \cdots, z_n) := \begin{cases} \left(\frac{1}{n}(z_1^k + \cdots + z_n^k)\right)^{\frac{1}{k}} & \text{if } k \neq 0, \\ (z_1 \cdots z_n)^{\frac{1}{n}} & \text{if } k = 0. \end{cases}$$

Then $|P_k| \leq |P_0|$ if $k \geq 0$ and $|P_k| \geq |P_0|$ if $k \leq 0$.

Proof. Note that $|P_0| = \rho$. We prove the inequality for $k > 0$. The case $k < 0$ follows analogously. By the Triangle Inequality, we have that

$$|P_k| \leq \left(\frac{1}{n}\sum_{i=1}^{n} |z_i^k|\right)^{\frac{1}{k}} = \left(\frac{1}{n}\sum_{i=1}^{n} \rho^k\right)^{\frac{1}{k}} = \rho\,.$$

This finishes the proof. □

It would be tempting to say that $k \mapsto |P_k|$ is a decreasing function for all non-zero $z_1, \ldots, z_n$ with a constant modulus ρ. It turns out that this is not true. However, the following theorem shows that under some extra conditions, we can still obtain an analogous result to that in Theorem 4.

Theorem 8. *Let $z_1, \in \mathbb{C}$, both with modulus $\rho > 0$ and $-\pi \leq \arg(z_1) - \arg(z_2) \leq \pi$. Let P_k be defined as above. Then*

$$|P_r(z_1, z_2)| \leq |P_s(z_1, z_2)|$$

whenever $-1 \leq s \leq r \leq 1$. Equality holds if and only if $r = s$ or $z_1 = z_2 = \rho$. Furthermore, the interval $[-1, 1]$ is strict.

Proof. Set $z_1 = \rho e^{i\theta_1}$ and $z_2 = \rho e^{i\theta_2}$. A simple algebraic manipulation yields

$$|P_k| = |P_k(z_1, z_2)| = \begin{cases} \rho \Big| \cos(\frac{1}{2}k\mu) \Big|^{\frac{1}{k}} & \text{if } k \neq 0, \\ \rho & \text{if } k = 0, \end{cases}$$

with $\mu = \theta_1 - \theta_2 \in [-\pi, \pi]$ by our assumption. Note that since $\cos$ is even, it suffices to analyse $|P_k|$ when $\mu \in [0, \pi]$. For $k \in [-1, 1] \setminus \{0\}$ and $\mu \in [0, \pi]$, we have that $\cos(\frac{1}{2}k\mu) \geq 0$, so the absolute value sign may be dropped. Since $x \mapsto e^x$ is strictly increasing, our claim is equivalent to showing that the function $g : [-1, 1] \to \mathbb{R}$ defined by

$$g(k) = \begin{cases} \frac{1}{k} \ln \cos(\frac{1}{2}k\mu) & \text{if } k \neq 0, \\ 0 & \text{if } k = 0 \end{cases}$$

is non-increasing.

To do this, we first establish that the function $h : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ defined by

$$h(u) = -u \tan u - \ln \cos u$$

satisfies $h(u) \leq 0$, with equality if and only if $u = 0$. Indeed, $h(0) = 0$ and $h'(u) = -\tan u - u \sec^2 u + \tan u = -u \sec^2 u$. Since $\sec^2 u > 0$ for all $u \in (-\frac{\pi}{2}, \frac{\pi}{2})$, the sign of $h'(u)$ just depends on the sign of u. We have that h is strictly decreasing on $(0, \frac{\pi}{2})$ and strictly increasing on $(-\frac{\pi}{2}, 0)$. Hence, $h(u) \leq 0$ on the entire domain, with equality only at $u = 0$.

If $k \in [-1, 1] \setminus \{0\}$, then $g'(k)$ exists with

$$g'(k) = \frac{1}{k^2}\left(-\frac{k\mu}{2} \tan\left(\frac{k\mu}{2}\right) - \ln \cos\left(\frac{k\mu}{2}\right) \right) = \frac{1}{k^2} h\left(\frac{1}{2}k\mu\right) \leq 0\,.$$

Moreover, using the Maclaurin expansion $\ln \cos(x) = -\frac{1}{2}x^2 - \frac{1}{12}x^4 - \cdots$, we obtain that g is differentiable at $k = 0$ with

$$g'(0) = \lim_{k \to 0} \frac{g(k) - g(0)}{k} = \lim_{k \to 0} \frac{1}{k^2} \ln \cos\left(\frac{k\mu}{2}\right) = \lim_{k \to 0} \frac{-\frac{k^2\mu^2}{8} + O(k^4)}{k^2} = -\frac{\mu^2}{8} \leq 0\,.$$

Since $g'(k) \le 0$ for all $k \in [-1,1]$, it follows that g is non-increasing on $[-1,1]$, and therefore, if $-1 \le s \le r \le 1$, then

$$|P_r(a_1,a_2)| \le |P_s(a_1,a_2)|.$$

Equality holds if and only if g is constant on (s,r), which requires $r=s$ or $\mu=0$. The latter case is equivalent to $a_1=a_2=\rho$.

It remains to show that the interval $[-1,1]$ is strict. Assume first by means of contradiction that there exists $r>1$ such that

$$\rho\Big|\cos\Big(\frac{1}{2}r\mu\Big)\Big|^{\frac{1}{r}} = |P_r(a_1,a_2)| < |P_1(a_1,a_2)| = \rho\Big|\cos\Big(\frac{1}{2}\mu\Big)\Big|$$

for all $\mu \in [0,\pi]$. Set $\frac{1}{2}\mu = \frac{1}{2}\pi - x$ for $0 < x < \frac{1}{2}\pi$ and define the function $R:(0,\frac{1}{2}\pi)\to\mathbb{R}$ by

$$R(x) = \frac{\big|\cos(r(\frac{1}{2}\pi - x))\big|^{\frac{1}{r}}}{\cos(\frac{1}{2}\pi - x)} = \frac{\big|\cos(r(\frac{1}{2}\pi - x))\big|^{\frac{1}{r}}}{\sin x}.$$

Consider the behaviour of R as $x \to 0^+$. If r is not an odd integer, then as $x \to 0^+$, the numerator approaches a finite positive number while the denominator approaches 0. Hence, $R(x)\to\infty$ as $x\to 0^+$. Otherwise, r is an odd integer. In this case, $r \ge 3$, the numerator is $|\sin(rx)|^{\frac{1}{r}}$ and

$$|R(x)| = \left|\frac{\sin^{\frac{1}{r}}(rx)}{\sin x}\right| = r^{\frac{1}{r}} x^{\frac{1}{r}-1}\left|\frac{\left(\frac{\sin rx}{rx}\right)^{\frac{1}{r}}}{\frac{\sin x}{x}}\right|.$$

This again means $R(x)\to\infty$ as $x\to 0^+$.

By the definition of limit, there exists $x_0 > 0$ such that $|P_r(a_1,a_2)| > |P_1(a_1,a_2)|$ where $|\arg(a_1)-\arg(a_2)| = \pi - 2x_0$. This contradicts our assumption and we must have $r \le 1$.

Next, assume that there exists some $s < -1$ such that $|P_{-1}(a_1,a_2)| < |P_s(a_1,a_2)|$ for all $\mu\in[0,\pi]$. By letting $s=-k$ for some $k>1$ and using the fact that $x \mapsto \cos x$ is an even function, it can be seen that the resulting inequality is $|P_k| \le |P_1|$ for $k>1$ which is not true using the argument above. Again, this is a contradiction and we must have $s \ge -1$. This proves that the interval $[-1,1]$ is strict, which completes the proof. □

The above theorem shows that generalising the Power Mean Inequality for complex numbers is not easy, even with a constant modulus. Analogous relationships between three or more complex numbers are quite difficult to derive and have been omitted in this article. The result above is quite striking in the sense that it apparently challenges well-known and classical inequalities. We conjecture that such "anomalies" should hold for other well known and classical inequalities as well.

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