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Approximating graphical minimal surfaces through mean curvature flow

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1 Introduction

Geometric flows are a class of processes that describe the evolution of geometric objects in time. These flows are often driven by intrinsic or extrinsic characteristics, and are often used to simplify complex geometric shapes. One of the most studied geometric flows is the mean curvature flow, where the time evolution of the surface at each point is given by

$$\left(\frac{\partial \mathcal{M}_t}{\partial t}\right)^{\perp} = H\nu \,,$$

where H is the mean curvature and ν is an evolving Gauss map. It is shown in Section 2 that such an evolution yields a monotone non-increasing area, implying that the surfaces tend toward minimality—a stationary point of the first variation of area— under the mean curvature flow. Thus, it becomes natural to explore the use of mean curvature flow as a numerical tool for approximating minimal surfaces, which are solutions to the classic Plateau's Problem [2] – surfaces that can be parametrised by

$$\mathcal{M}\big(x, y, f(x,y)\big) \,,$$

with a graphically defined Dirichlet boundary condition.

While theoretical underpinnings for mean curvature flow are well-established, analytical solutions often remain intractable due to the nature of non-linear partial differential equations. However, for graphically defined surfaces, mean curvature flow simplifies to a non-linear parabolic partial differential equation, suggesting numerical approximation as a useful tool for estimating solutions.

This paper references the work of Sawyer Jacobson in [1], by presenting the derived algorithm for estimating graphical minimal surfaces through the use of mean curvature flow. We then provide simulations, demonstrating the effectiveness of the numerical approach by including scenarios with planar and rotationally symmetric boundary conditions, as well as the estimation of minimal surfaces for complex boundary data–where analytical solutions would remain enigmatic. The results provide evidence for previously established results in graphical minimal surface theory are true, including the inheritance of planarity and symmetry from a boundary. They also highlight the usefulness of graphical mean curvature flow in estimating minimal surfaces, while also providing a visual representation of their evolution throughout the flow.

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2 Algorithm

Take a family of surfaces parameterised by

$$\mathcal{M}(u_1, u_2, s) = F(u_1, u_2) + s \left\langle \frac{\partial F_s}{\partial s}, \nu \right\rangle \nu(u_1, u_2)\,,$$

where s is a compactly supported variational map and ν is a unit normal vector. Now, this family of surfaces has a first variation of area given by

$$\frac{d}{ds}\big[\mathcal{M}(s)\big]\bigg|_{s=0} = -\int_{\mathcal{M}} \left\langle \frac{\partial F_s}{\partial s}, \nu \right\rangle H dA\,,$$

where H is the mean curvature. So, we choose F_s such that

$$\left\langle \frac{\partial F_s}{\partial s}, \nu \right\rangle = H(F_s)\,,$$

yielding the monotone non-increasing

$$\frac{d}{ds}\big[\mathcal{M}(s)\big]\bigg|_{s=0} = -\int_{\mathcal{M}} H^2 dA\,,$$

with equilibrium at $H = 0$. This motivates our definition of mean curvature flow.

Definition 1 (Mean Curvature Flow)**.** *Given a family of surfaces $\{\mathcal{M}_t\}$, the mean curvature flow is the evolution of the surface by the system of equations*

$$\begin{cases} \left(\dfrac{\partial \mathcal{M}_t}{\partial t}\right)^{\perp} = H\nu\,; \\ \mathcal{M}_0 = \mathcal{M}\,. \end{cases}$$

Now, it was shown in [1] that a minimal surface $\mathcal{M}$ parameterised by $\mathcal{M}\big(x, y, f(x, y)\big)$ must satisfy the partial differential equation

$$\frac{\partial}{\partial x}\left(\frac{f_x}{\sqrt{1 + |\nabla f|^2}}\right) + \frac{\partial}{\partial y}\left(\frac{f_y}{\sqrt{1 + |\nabla f|^2}}\right) = 0\,.$$

So, in pair with the Dirichlet boundary condition, we get that a graphical solution to Plateau's Problem in Ω is given by the system of equations

$$\begin{cases} \operatorname{div}\dfrac{\nabla f}{\sqrt{1 + |\nabla f|^2}} = 0 \quad \text{in } \Omega\,; \\ \mathcal{M}(x, y, t) = \Gamma(x, y) \quad \text{on } \partial\Omega\,. \end{cases}$$

As our mean curvature flow is monotone non-increasing, flows will generally tend towards minimality[3], implying the usefulness of such a flow to estimate minimal surfaces. In our graphical case, the mean curvature flow evolution simplifies to the system of equations

$$\begin{cases} \dfrac{\partial f}{\partial t} = \sqrt{1+|\nabla f|^2}\,\mathrm{div}\dfrac{\nabla f}{\sqrt{1+|\nabla f|^2}} & \text{in } \Omega \times (0,\infty) \\ f(x,y,t) = \Gamma(x,y) & \text{on } \partial\Omega \times (0,\infty) \\ f(x,y,0) = \rho(x,y) & \text{in } \bar{\Omega} \times \{0\} \end{cases}$$

for some function $\rho(x,y) : \Omega \longrightarrow \mathbb{R}$ such that $\rho = \Gamma$ on $\partial\Omega$.

3 Simulations

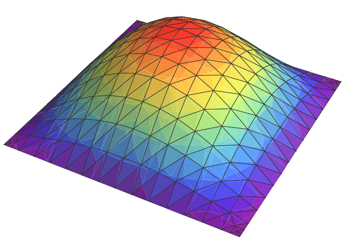
Now that we have a way to estimate graphical minimal surfaces, we aim to simulate multiple cases. *All simulations and visualisations were implemented in Mathematica.*

3.1 Inheritance of planarity

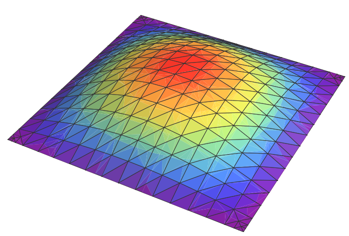
In [1], it was shown that graphical minimal surfaces $\mathcal{M} \subseteq \mathbb{R}^3$ with boundary $\Gamma \subseteq \pi$, for some plane π, will be such that $\mathcal{M} \subseteq \pi$. Therefore, we look to simulate such cases for $\mathcal{M}_t, t \in [0,T)$. In this simulation, we use the initial surface data

$$f(0,x,y) := \sin\left(\frac{\pi x}{5}\right)\sin\left(\frac{\pi y}{5}\right).$$

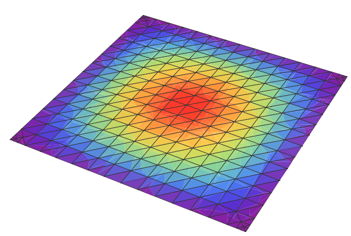
We also fix a uniformly-zero Dirichlet boundary condition on $[0,5] \times [0,5]$. Then, we let the surface flow, governed by the partial differential equation in Section 2. We see, as expected, a smoothing effect, as the surface tends towards its planar boundary.



(a) Initial Data



(b) Intermediate Data



(c) Final Data

As expected, $\mathcal{M}_t$ converged to a subset of π, with $\partial\mathcal{M}_t = \Gamma \subseteq \pi$, exhibiting the property shown in [1].

[3] A counterexample occurs when a surface yields singularities.

3.2 Inheritance of rotational symmetry

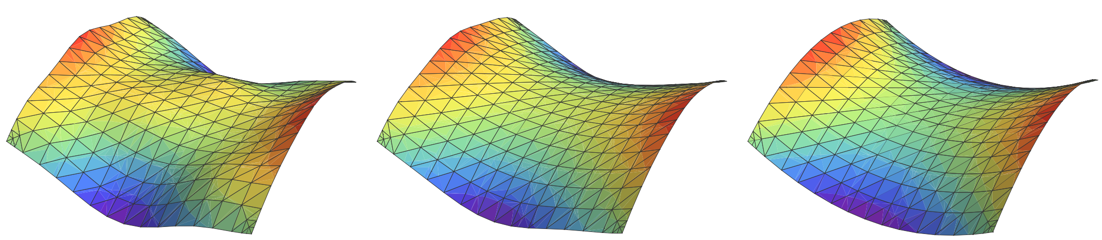
It was also shown in [1] that graphical minimal surfaces $\mathcal{M} \subseteq \mathbb{R}^3$, a rotationally symmetric boundary Γ, induces rotational symmetry on $\mathcal{M}$. It is therefore interesting to simulate such cases. When the boundary Γ can be embedded in a plane, this property is trivial as all planes are symmetric. So, we take the non-planar boundary data

$$\Gamma = 0.1(x-2.5)^2 - 0.1(y-2.5)^2 + 0.8\,,$$

which represents the boundary of a hyperbolic paraboloid in $\mathbb{R}^3$. We also must define the initial boundary data, which we choose to be non-symmetric to exhibit how the symmetry of the boundary truly imposes such a property on $\mathcal{M}$:

$$\begin{aligned} f(0,x,y) := {} & 0.1(x-2.5)^2 - 0.1(y-2.5)^2 + 0.8 \\ & + 0.25e^{-(x-3.0)^2-(y-3.2)^2} + 0.2e^{-(x-1.5)^2-(y-2.0)^2} + 0.15e^{-(x-4.0)^2-(y-1.0)^2} \\ & + 0.1\sin(2x)\cos(1.2y) + 0.05\sin(3y)\cos(1.5x)\,. \end{aligned}$$

After a short period of time, we see that the surface begins to evolve towards symmetry. Finally, it converges to the hyperbolic paraboloid, spanning the boundary defined above As expected, the surface $\mathcal{M}$ exhibits two-fold rotational symmetry, just as does



(a) Initial Data (b) Intermediate Data (c) Final Data

its boundary.

3.3 Complex boundary data

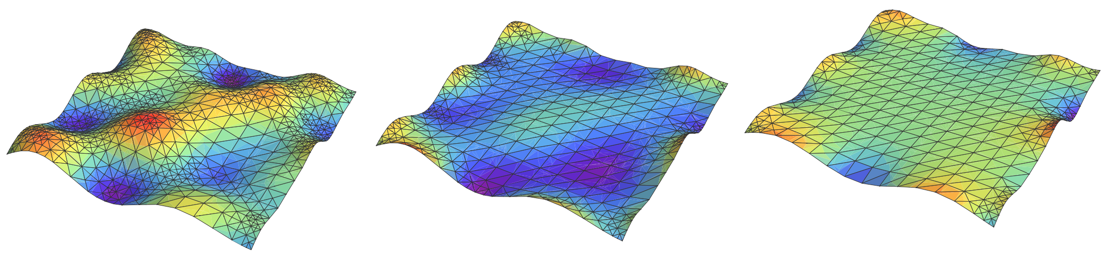
One of the primary applications of graphical mean curvature flow is to estimate minimal surfaces with complicated boundary data. Therefore, we apply our algorithm to simulate such cases under multiple boundary conditions. So, we define a complicated boundary:

$$\begin{aligned} \Gamma := {} & 0.2\sin(2x)\cos(1.5y) + 0.15\sin(3y)\cos(1.2x) + 0.05\sin(xy) + 0.5 \\ & + 0.1e^{-(x-2)^2-(y-3)^2} - 0.08e^{-(x-3.5)^2-(y-1.5)^2} \end{aligned}$$

with initial condition

$$
\begin{aligned}
f(0,x,y) := {} & 0.2\sin(2x)\cos(1.5y) + 0.15\sin(3y)\cos(1.2x) + 0.05\sin(xy) \\
& + 0.1\sin(2.5x)\cos(1.8y) + 0.05\sin(3y)\cos(2x) + 0.5 \\
& + 0.1e^{-(x-2)^2-(y-3)^2} - 0.08e^{-(x-3.5)^2-(y-1.5)^2} + 0.2e^{-(x-1.2)^2-(y-1.5)^2} \\
& + 0.15e^{-(x-3.5)^2-(y-3)^2} + 0.1e^{-(x-2.5)^2-(y-2.5)^2}.
\end{aligned}
$$

This is analytically difficult to solve, highlighting the importance of our algorithm in estimating such surfaces.



(a) Initial Data (b) Intermediate Data (c) Final Data

We see that the surface flows, eventually converging, giving us an estimate of the minimal spanning surface for the given boundary condition.

References

[1] S. Jacobson, On graphical minimal surfaces and the homotheticity of singular behavior in mean curvature flow, *Zenodo*, DOI 10.5281/zenodo.16903054, 2025.

[2] Wikipedia, Plateau's problem, last accessed on 2026-02-24.