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Sum and square sum of consecutive tetrahedral numbers via the Cauchy Residue Theorem

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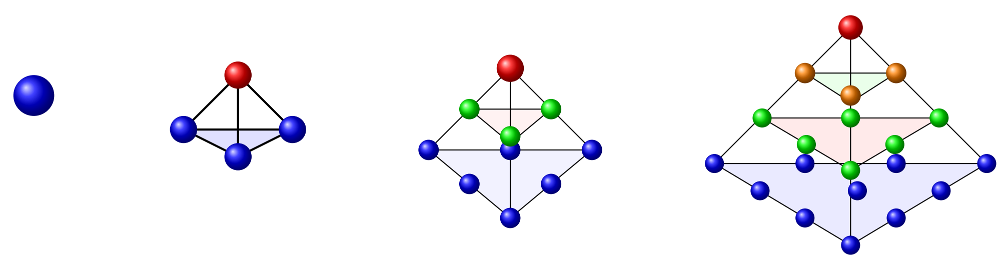
1 Introduction

The tetrahedral number [1] is a special type of figurate number that represents how many objects can be arranged in the form of a tetrahedron, a pyramid with a triangular base and three sides. These numbers show how many items are needed to build up this shape layer-by-layer, starting from a single point and adding triangular layers on top of each other. The nth tetrahedral number can be defined as the sum of first n triangular numbers $T_n = \frac{n(n+1)}{2}$ as follows:

$$\tau_n = T_1 + T_2 + T_3 + \cdots + T_n = 1 + 3 + 6 + \cdots + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}\,.$$

This means the first few tetrahedral numbers are $1, 4, 10, 20, 35, \ldots$, corresponding to the triangular numbers $1, 3, 6, 10, 15$. The tetrahedral numbers are shown geometrically by the following figure:

Figure: Geometric Representation of Tetrahedral Numbers



$\tau_1 = 1$ $\quad$ $\tau_2 = 1 + 3 = 4$ $\quad$ $\tau_3 = 1 + 3 + 6 = 10$ $\quad$ $\tau_4 = 1 + 3 + 6 + 10 = 20$

The figure illustrates how the tetrahedron corresponding to the nth $(n = 1, 2, 3, \ldots)$ tetrahedral number consists of n triangular layers with $i(i+1)/2$ dots in the ith layer $(i = 1, 2, 3, \ldots, n)$. Several researchers [2, 3, 4] have made significant contributions to the different aspects of tetrahedral numbers, including power sum identities. Recently,

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Chakraborty and Mortini [6] presented a nice derivation of the identity of sums of squares of consecutive natural numbers using the contour integral. Motivated by this work, we here establish an alternative proof of the sum and square sum identities for consecutive tetrahedral numbers using the Cauchy Residue Theorem [5]. Before going to derive the identities on tetrahedral numbers, we discuss the Cauchy Residue Theorem for complex integration in the following section.

2 Cauchy Residue Theorem

Consider a complex function $f(z)$ that has an isolated singularity at a point z_0. This means that $f(z)$ is not analytic at z_0 but is analytic everywhere else in some small neighbourhood around z_0 (a "punctured disk"). In such a region, $f(z)$ can be represented by a Laurent series expansion:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots .$$

The part with the negative powers of $z - z_0$ is called the principal part. The residue of $f(z)$ at z_0, denoted $\mathrm{Res}(f, z_0)$, is defined as the coefficient of the term $(z - z_0)^{-1}$ in this Laurent series:

$$\mathrm{Res}(f, z_0) = b_1 .$$

Intuitively, the residue captures the "strength" or the "non-analytic part" of the function's behaviour at that specific singularity. Now, we shall state the powerful theorem that connects global integration with local properties (residues).

Proposition 1 (Cauchy's Residue Theorem)**.** [5] *Let C be a simple, closed, positively-oriented (counter-clockwise) contour within a simply connected domain D. Let $f(z)$ be a function that is analytic on and inside C, except for a finite number of isolated singular points $z_1, z_2, \ldots, z_n$ located inside C.*

Then, the contour integral of $f(z)$ around C is given by

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_k) .$$

The value of the integral around a closed curve is simply $2\pi i$ times the sum of the residues of the function at all singularities enclosed by the curve.

3 Two important identities on tetrahedral numbers

In this section, identities concerning the sum and square sum of consecutive tetrahedral numbers are proved by complex-analysis methods.

Proposition 2. [7] *For the tetrahedral numbers $\tau_i = \frac{i(i+1)(i+2)}{6}$,*

$$\tau_1 + \tau_2 + \cdots + \tau_n = \frac{n(n+1)(n+2)(n+3)}{24}\,.$$

Proof. Let us consider the function $f_k(z) = \frac{(1+z)^{k+2}}{z^4}$. Now, consider the finite sum

$$\begin{aligned}
\sum_{k=1}^{n} f_k(z) &= \sum_{k=1}^{n} \frac{(1+z)^{k+2}}{z^4} \\
&= \frac{1}{z^4} \sum_{k=1}^{n} (1+z)^{k+2} \\
&= \frac{(1+z)^3}{z^4} \sum_{k=1}^{n} (1+z)^{k-1} \\
&= \frac{(1+z)^3}{z^4} \cdot \left(\frac{(1+z)^n - 1}{1+z-1} \right) \\
&= \frac{(1+z)^{n+3} - (1+z)^3}{z^5}\,.
\end{aligned}$$

Thus,

$$\sum_{k=1}^{n} f_k(z) = F(z), \quad \text{where } F(z) = \frac{(1+z)^{n+3} - (1+z)^3}{z^5}\,. \tag{1}$$

Let C be the contour $|z| = 1$. By integrating both sides of (1) along C, and interchanging the sum and integral (justified by the linearity of integration over a finite sum), we get

$$\sum_{k=1}^{n} \oint_C f_k(z)\,dz = \oint_C F(z)\,dz\,.$$

The only singularity of both $f_k(z)$ and $F(z)$ inside C is at $z = 0$, which is a pole of order 4 for $f_k(z)$ and order 5 for $F(z)$. We now apply the Cauchy Residue Theorem (Proposition 2) to *each individual integral* in Equation (3).

Applying the theorem to the left-hand side, we get

$$\oint_C f_k(z)\,dz = 2\pi i \operatorname{Res}(f_k, 0)\,.$$

Substituting this into (3) gives

$$\sum_{k=1}^{n} 2\pi i \operatorname{Res}(f_k, 0) = \oint_C F(z)\,dz\,. \tag{2}$$

Applying the theorem to the integral on the right-hand side of (2) gives

$$\oint_C F(z)\,dz = 2\pi i \operatorname{Res}(F, 0)\,.$$

Substituting this back into (2) yields

$$\sum_{k=1}^{n} 2\pi i \operatorname{Res}(f_k, 0) = 2\pi i \operatorname{Res}(F, 0)\,.$$

Cancelling the common factor $2\pi i$ (nonzero) gives

$$\sum_{k=1}^{n} \operatorname{Res}(f_k, 0) = \operatorname{Res}(F, 0)\,. \tag{3}$$

We now compute the residues. By Appendix A1, $\operatorname{Res}(f_k, 0) = \binom{k+2}{3} = \frac{(k+2)(k+1)k}{6} = \tau_k$. From Appendix A2, we have $\operatorname{Res}(F, 0) = \binom{n+3}{4} = \frac{(n+3)(n+2)(n+1)n}{24}$. Substituting these identities into Equation (3) gives

$$\sum_{k=1}^{n} \tau_k = \frac{n(n+1)(n+2)(n+3)}{24}\,.$$

This completes the proof. □

Remark 3. [7] The result of Proposition 2 is a specific case of a more general pattern. The triangular number $T_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ is the sum of the first n positive integers. The tetrahedral number $\tau_n = \sum_{i=1}^{n} T_i$ is the sum of the first n triangular numbers. This process can be iterated, and the m-th iterated sum (or the m-th dimensional simplex number) is given by the Hockey-stick identity, yielding

$$\sum_{i_1=1}^{n} \sum_{i_2=1}^{i_1} \cdots \sum_{i_m=1}^{i_{m-1}} i_m = \frac{n(n+1)\cdots(n+m-1)}{m!} = \binom{n+m-1}{m}\,.$$

This is a known result, often derived via telescoping series or combinatorial reasoning [7]. Proposition 2 corresponds to the case $m = 3$. A proof for the general case using the contour integral method presented here would involve considering functions of the form $\frac{(1+z)^{k+m}}{z^{m+1}}$ and following similar steps.

In the next proposition, we present the formula for the sum of squares of consecutive tetrahedral numbers. This identity has already been derived by several approaches, including using power sum identities of natural numbers [8]. However, no one has derived it using complex integration as far as we know.

Proposition 4. *For n-th tetrahedral number $\tau_n = \frac{n(n+1)(n+2)}{6}$,*

$$\tau_1^2 + \tau_2^2 + \cdots + \tau_n^2 = \binom{n+3}{4} + 12\binom{n+3}{5} + 30\binom{n+3}{6} + 20\binom{n+3}{7}\,.$$

Proof. Let us consider the function

$$g_k(z) = \frac{(z^3 + 12z^2 + 30z + 20)(1+z)^{k+2}}{z^7}$$

and consider the finite sum

$$\begin{aligned}
\sum_{k=1}^{n} g_k(z) &= \frac{(z^3 + 12z^2 + 30z + 20)}{z^7} \sum_{k=1}^{n} (1+z)^{k+2} \\
&= \frac{(z^3 + 12z^2 + 30z + 20)}{z^7} (1+z)^3 \sum_{k=1}^{n} (1+z)^{k-1} \\
&= \frac{(z^3 + 12z^2 + 30z + 20)}{z^7} (1+z)^3 \left(\frac{(1+z)^n - 1}{(1+z) - 1} \right) \\
&= \frac{(z^3 + 12z^2 + 30z + 20)\left[(1+z)^{n+3} - (1+z)^3\right]}{z^8} = G(z).
\end{aligned}$$

Thus, we have

$$\sum_{k=1}^{n} g_k(z) = G(z)\,. \tag{4}$$

Let C be the contour $|z| = 1$. By integrating both sides of (4) along C, and interchanging the sum and integral, we get

$$\sum_{k=1}^{n} \oint_C g_k(z)\, dz = \oint_C G(z)\, dz\,. \tag{5}$$

The only singularity of both $g_k(z)$ and $G(z)$ inside C is at $z = 0$, which is a pole of order 7 for $g_k(z)$ and order 8 for $G(z)$. We apply the Cauchy Residue Theorem to each integral.

For the left-hand side of (5),

$$\oint_C g_k(z)\, dz = 2\pi i \operatorname{Res}(g_k, 0)\,,$$

so

$$\sum_{k=1}^{n} 2\pi i \operatorname{Res}(g_k, 0) = \oint_C G(z)\, dz\,. \tag{6}$$

For the right-hand side of (6),

$$\oint_C G(z)\, dz = 2\pi i \operatorname{Res}(G, 0)\,.$$

Substituting this into (6) gives

$$\sum_{k=1}^{n} 2\pi i \operatorname{Res}(g_k, 0) = 2\pi i \operatorname{Res}(G, 0)\,.$$

Therefore,

$$\sum_{k=1}^{n} \operatorname{Res}(g_k, 0) = \operatorname{Res}(G, 0)\,. \tag{7}$$

We now compute the residues. By Appendix A3, $\operatorname{Res}(g_k, 0) = \frac{k^2(k+1)^2(k+2)^2}{36} = \tau_k^2$. From Appendix A4, we have $\operatorname{Res}(G, 0) = 20\binom{n+3}{7} + 30\binom{n+3}{6} + 12\binom{n+3}{5} + \binom{n+3}{4}$. Substituting these identities into (7) gives

$$\sum_{k=1}^{n} \tau_k^2 = \binom{n+3}{4} + 12\binom{n+3}{5} + 30\binom{n+3}{6} + 20\binom{n+3}{7}\,.$$

This completes the proof. □

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Appendices

A 1. Residue of $f_k(z) = \frac{(1+z)^{k+2}}{z^4}$ at $z = 0$

We have

$$f_k(z) = \frac{(1+z)^{k+2}}{z^4}\,.$$

The residue is the coefficient of z^{-1} in the Laurent expansion, which equals the coefficient of z^3 in $(1+z)^{k+2}$. Expanding $(1+z)^{k+2}$, we get

$$(1+z)^{k+2} = \sum_{j=0}^{\infty} \binom{k+2}{j} z^j\,.$$

The residue is the coefficient of z^3, namely

$$\operatorname{Res}(f_k, 0) = [z^3](1+z)^{k+2} = \binom{k+2}{3} = \frac{(k+2)(k+1)k}{6}\,.$$

A 2. Residue of $F(z) = \frac{(1+z)^{n+3}-(1+z)^3}{z^5}$ at $z = 0$

We have

$$F(z) = \frac{(1+z)^{n+3} - (1+z)^3}{z^5}\,.$$

The residue is the coefficient of z^{-1}, which equals the coefficient of z^4 in $(1+z)^{n+3} - (1+z)^3$. By expanding both of these terms, we get

$$(1+z)^{n+3} - (1+z)^3 = \sum_{j=0}^{\infty} \binom{n+3}{j} z^j - \left(1 + 3z + 3z^2 + z^3\right)$$

For $j = 4$, the coefficient of z^j is

$$[z^4]\left[(1+z)^{n+3} - (1+z)^3\right] = \binom{n+3}{4} - 0 = \binom{n+3}{4}$$

since $(1+z)^3$ has no z^4 term. The residue is therefore

$$\operatorname{Res}(F, 0) = \binom{n+3}{4} = \frac{1}{24}(n+3)(n+2)(n+1)n\,.$$

A 3. Residue of $g_k(z)$ at $z = 0$

We have

$$g_k(z) = \frac{(z^3 + 12z^2 + 30z + 20)(1+z)^{k+2}}{z^7} .$$

Let $P(z) = z^3 + 12z^2 + 30z + 20$. The residue is the coefficient of z^{-1} in $g_k(z)$, which equals the coefficient of z^6 in $P(z)(1+z)^{k+2}$. The binomial expansion of $(1+z)^{k+2}$ is

$$(1+z)^{k+2} = \sum_{j=0}^{\infty} \binom{k+2}{j} z^j ,$$

so

$$g_k(z) = P(z)(1+z)^{k+2} = \big(20 + 30z + 12z^2 + z^3\big) \sum_{j=0}^{\infty} \binom{k+2}{j} z^j .$$

Thus, the coefficient of z^6 in $g_k(z)$ is

$$20\binom{k+2}{6} + 30\binom{k+2}{5} + 12\binom{k+2}{4} + \binom{k+2}{3} .$$

Hence,

$$\mathrm{Res}(g_k, 0) = 20\binom{k+2}{6} + 30\binom{k+2}{5} + 12\binom{k+2}{4} + \binom{k+2}{3} = \frac{k^2(k+1)^2(k+2)^2}{36} .$$

A 4. Residue of $G(z)$ at $z = 0$

We have

$$G(z) = \frac{(z^3 + 12z^2 + 30z + 20)\left[(1+z)^{n+3} - (1+z)^3\right]}{z^8} .$$

Let $P(z) = z^3 + 12z^2 + 30z + 20$ and $Q(z) = (1+z)^{n+3} - (1+z)^3$. Expanding $(1+z)^{n+3}$, we get

$$(1+z)^{n+3} = \sum_{j=0}^{n+3} \binom{n+3}{j} z^j,$$

and

$$(1+z)^3 = 1 + 3z + 3z^2 + z^3.$$

Thus, expanding $Q(z)$, we get

$$Q(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \sum_{j=4}^{n+3} a_j z^j ,$$

where

$$a_0 = \binom{n+3}{0} - 1 = 0 , \quad a_1 = \binom{n+3}{1} - 3 = n , \quad a_2 = \binom{n+3}{2} - 3 , \quad a_3 = \binom{n+3}{3} - 1 ,$$

and $a_j = \binom{n+3}{j}$ for $j \geq 4$. The residue of G at 0 is the coefficient of z^7 in

$$P(z)Q(z) = (20 + 30z + 12z^2 + z^3) \sum_{j \geq 0} a_j z^j \, ,$$

namely

$$20a_7 + 30a_6 + 12a_5 + 1 \cdot a_4 \, .$$

Since $a_j = \binom{n+3}{j}$ for $j \geq 4$, we have

$$\mathrm{Res}(G, 0) = 20\binom{n+3}{7} + 30\binom{n+3}{6} + 12\binom{n+3}{5} + \binom{n+3}{4} \, .$$