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Many aspects of exponential functions

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1 Introduction

In this article, we study various aspects of exponential functions. Many interesting stories and a discovery are presented.

It is well known that exponential functions play a crucial role in explaining various scientific and social phenomena worldwide. For example, you can find many interesting facts on exponential functions in Kit Yates's fascinating book "The Math of Life and Death" [19]. When the authors of the present article read his book in a high school elective class, they began to look for interesting facts about exponential functions. Most of the facts in this article have already been discovered, but they are new and interesting to many readers.

In Section 3, we study the exponential function that describes increasing populations, and the logistic equation that describes the increasing population of creatures and the process that stops the increase. We also study the SIR model. Next, we study examples of decreasing exponential functions. In Section 4, we examine the forgetting curve, a graph that illustrates the mathematical formula describing the process of forgetting. This forgetting curve is applied to software for memorising. In Section 5, we study the formula that describes the recovery process of heart rate after exercise. This formula is used to evaluate a person's cardiopulmonary function. Next, we study the inverse proportional function, which resembles the decreasing exponential function. In Section 6, we study the formula that describes the relationship between the psychological perception of time and ageing. Then, we study geometric progression. In Section 7, we study a historical story in which the exponential function plays a significant role. In Section 8, we study the formula that describes how news spreads.

Next, we study the power tower. The power tower increases far more rapidly than traditional exponential functions, and the calculation of the power tower is usually beyond the most powerful computers. However, power tower $\mod 10$ is a fascinating topic. The only new result in this article is the fastest algorithm for calculating the power tower $\mod 10$, presented in Section 9.

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To the best of our knowledge, this algorithm represents a new mathematical discovery. The problem of calculating the power tower modulo 10 is well-known and has been featured in some computer programming contests. This algorithm was presented by two of the authors of the present article in [10] in 2003 without any proof, and the authors present this algorithm with a rigorous mathematical proof for the first time, in Section 9.

2 What is an exponential function?

Here, we explain the exponential function. The functions that most students encounter for the first time in a class are linear, and after that, they learn quadratic functions. An example of a linear function is in Figure 1, and an example of a quadratic function is in Figure 2.

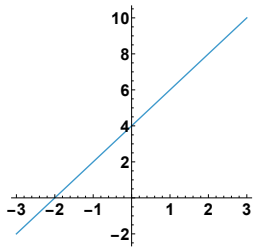

Figure 1: $y = 2x + 4$

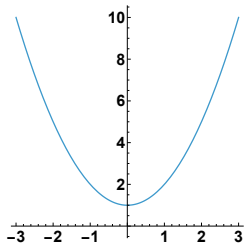

Figure 2: $y = x^2 + 1$

However, linear and quadratic functions are not among the most common functions in science and social science. The most common functions are exponential functions, and examples of exponential functions are Figures 3 and 4.

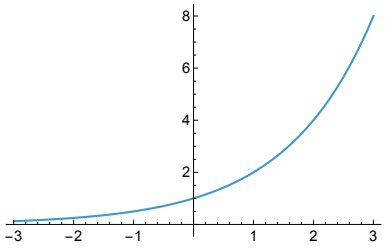

Figure 3: $y = 2^x$

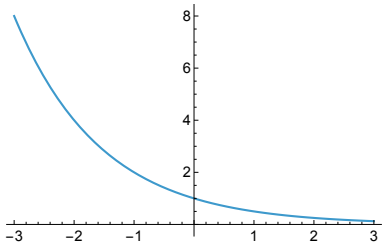

Figure 4: $y = (\frac{1}{2})^x$

In Figures 3 and 4, the range of x is from -3 to 3, and we need to study the larger and smaller values of x if we want to know the properties of exponential functions. For larger values of x, since $2^1 = 2$, $2^2 = 2 \times 2 = 4$, $2^3 = 2 \times 2 \times 2 = 8$, $2^4 = 2 \times 2 \times 2 \times 2 = 16$, $2^5 = 32$, $2^6 = 64$, $2^7 = 128$, $2^8 = 256$, $2^9 = 512$ and $2^{10} = 1024$, we get the graph in Figure 5. Then, by connecting discrete points, we get the graph in Figure 6.

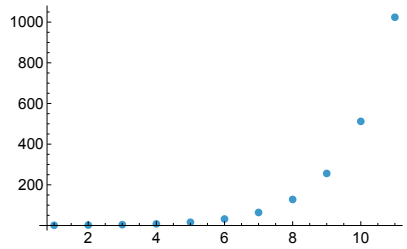

Figure 5: $y = 2^x$ for $x = 1, 2, \ldots, 10$

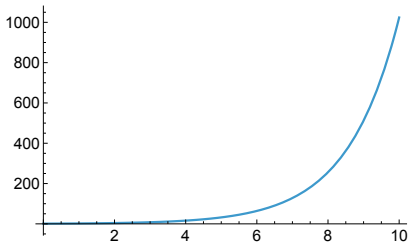

Figure 6: $y = 2^x$

For smaller values of x, since $(\frac{1}{2})^1 = \frac{1}{2}, (\frac{1}{2})^2 = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}, \ldots$, and $(\frac{1}{2})^{10} = \frac{1}{1024}$, we get the graph in Figure 7. Then, by connecting discrete points, we get the graph in Figure 8.

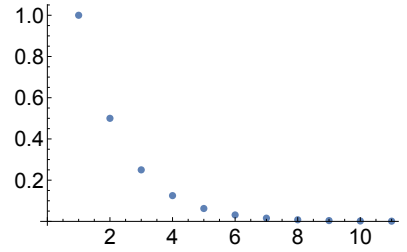

Figure 7: $y = 2^x$ for $x = 1, 2, \ldots, 10$

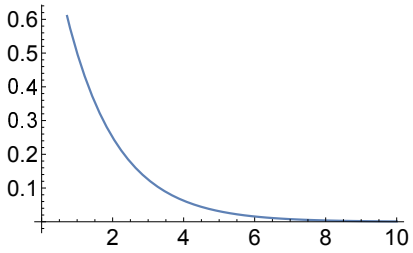

Figure 8: $y = 2^x$

The next theorem explains the properties presented in Figures 6 and 8.

Theorem 1.
(1) For $a > 1$, the exponential function $y = a^x$ satisfies the following properties:
(i) As x increases, $f(x)$ heads to infinity.
(ii) As x decreases, $f(x)$ heads to 0.
(2) For $a \in (0, 1)$, the exponential function $y = a^x$ satisfies the following properties:
(i) As x increases, $f(x)$ heads to 0.
(ii) As x decreases, $f(x)$ heads to infinity.

We do not prove Theorem 1, but (i) of (1) and (i) of (2) are obvious from Figures 6 and 8. As for (ii) of (1) and (ii) of (2), we need to understand exponents of negative integers, such as $2^0 = 1$, $2^{-1} = \frac{1}{2}$, $2^{-2} = \frac{1}{2^2}$, ... If you want to understand the exponential function precisely, then you need to know rational exponents such as $2^{\frac{1}{2}} = \sqrt[2]{2}$, $2^{\frac{1}{n}} = \sqrt[n]{2}$, and so on. As for a good explanation on rational exponents, see [1].

There is a very important number for exponential functions. That is the number

$$e = \lim_{n \to \infty} \left(\frac{1}{n} + 1 \right)^n . \tag{1}$$

The exponential function $y = e^x$ has a remarkable property presented in the following theorem.

Theorem 2. *For $y = e^x$,*

$$\frac{dy}{dx} = e^x.$$

Since this is a well-known theorem, we omit the proof. By Theorem 2, we obtain an interesting fact about the slope of exponential functions; see also [2].

3 Exponential functions and growth

Increasing exponential functions describe many phenomena in the world. One of the most well-known is the growth of microorganisms. The mechanism of development is simple. The first organism splits into two daughter organisms, which then each split to form four, and so on. A virus will spread exponentially. Exponential growth can occur in the world of physics. In a nuclear chain reaction, each uranium nucleus that undergoes fission produces multiple neutrons, each of which can be absorbed by adjacent uranium atoms, causing them to fission in turn. We have also witnessed exponential growth in the economy. Economic growth is typically expressed in percentage terms, and it is often exponential.

3.1 The Logistic Equation

The number of microorganisms in a culture will continue increasing exponentially until an essential nutrient for the microorganisms is exhausted. In reality, owing to limited environments and resources, the growth of microorganisms eventually encounters a brake. As the number of microorganisms increases, the growth rate decreases, and the population eventually reaches a point of saturation. The logistic equation incorporates this point and models the growth of microorganisms. The logistic equation is expressed as a differential equation:

$$\frac{dN}{dt} = rN\Big(1 - \frac{N}{K}\Big). \tag{2}$$

Here, N represents the number of microorganisms, t the time, and dN/dt the rate of population growth; r is the intrinsic growth rate, and K is a constant known as the *carrying capacity*. By solving Equation (2), we obtain

$$N = \frac{K}{1 + \big(\frac{K}{N_0} - 1\big)e^{-rt}}, \tag{3}$$

where N_0 is the number of microorganisms when $t = 0$. Equation (3) describes how the growth rate decreases as the number of microorganisms increases and approaches the environment's carrying capacity. Equation (3) can also be applied to human population growth or economic growth.

Example 3. *Here is a graph of the logistic function generated by the Mathematica code in Code A of the Appendix.*

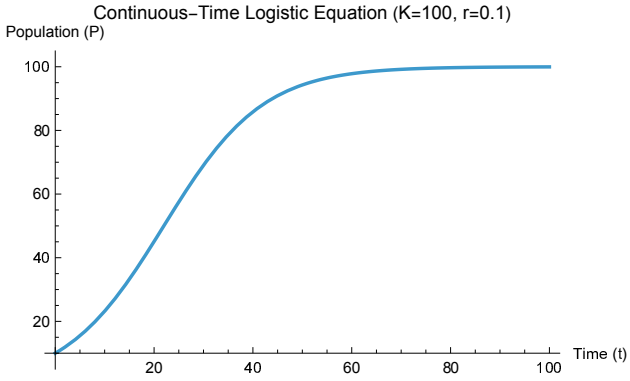

Figure 9: The logistic function

3.2 The SIR model

A virus will spread exponentially if no artificial or naturally acquired immunity is available. In reality, artificial or naturally acquired immunization is usually available. Kermack and McKendrick [9] proposed the SIR model that incorporates the effects of immunisation. In the SIR model, the entire population is divided into three segments: susceptible individuals (S), infected individuals (I) and recovered individuals (R). Susceptible individuals S transit to infected individuals I at a constant rate proportional to the product of the number of susceptible individuals S and the number of infected individuals I. Infected individuals I transit to recovered individuals R at a constant rate (the duration of infection follows an exponential distribution). This time evolution can be represented as a continuous dynamical system described by nonlinear ordinary differential equations as follows:

$$\begin{aligned}
\frac{dS}{dt}(t) &= \beta S(t)I(t)\,; \\
\frac{I}{dt}(t) &= \beta S(t)I(t) - \gamma I(t)\,; \\
\frac{R}{dt}(t) &= \gamma I(t)\,.
\end{aligned}$$

Example 4. *Here, we present a graph of the SIR model generated by the Mathematica code shown in Code B of the Appendix.*

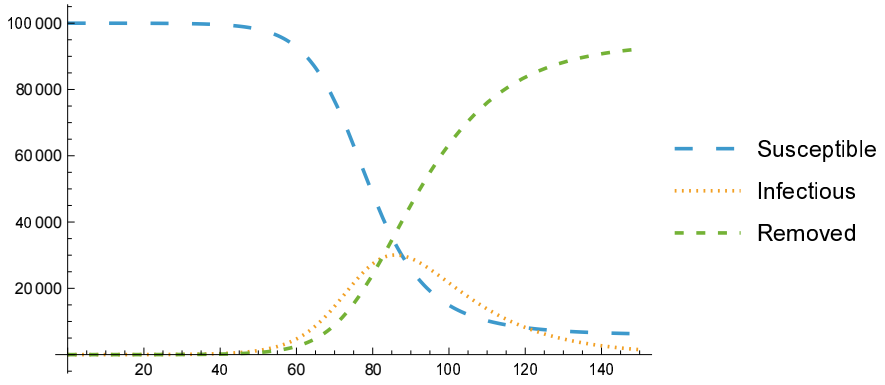

Figure 10: The SIR model

4 The forgetting curve

The first attempt to scientifically and quantitatively capture the highly complex and subjective phenomenon of human memory was made by the late 19th-century German psychologist Hermann Ebbinghaus [4]. His research was revolutionary in suggesting that the laws of memory might possess a mathematical rigour. He devised meaningless syllables composed of consonant-vowel-consonant patterns (e.g., DAX, ZUU, KUF) and adopted them as experimental materials. This allowed him, as the subject, to replicate the process of memorising entirely new information from scratch and to measure the pure rate of forgetting. He measured the time taken to memorise these syllable lists completely and the difference in time required to relearn them after a fixed period (e.g., 1 hour later, 1 day later). By using this savings rate in relearning time as an indicator of memory retention rate, he expressed memory decay as a quantitative value.

Ebbinghaus's graphical representation of these measurement results later became known as the *forgetting curve*. The characteristic decay pattern shown by the forgetting curve is elegantly modelled by a negative exponential function, which is used in natural sciences and engineering to describe decay phenomena. The memory decay can be mathematically expressed as

$$\frac{dR}{dt} = -kR\,,$$

where R is the retention rate (that is, the amount of memory over a period), dR/dt is the rate of change in memory over time, and k is a positive constant called the "forgetting rate". The negative sign indicates that the memory quantity R decreases with time. This differential equation means that the speed at which memory is forgotten is proportional to the amount currently remembered. From the above differential equation,

we obtain the following formula:

$$R(T) = R_0 e^{-kt}\,,$$

where $R(t)$ is the memory retention rate at time t; e is the base of the natural logarithm (approximately 2.718); R_0 is the initial memory amount and k is the decay rate (decay constant).

The decay rate is believed to vary based on the following factors, the first of which is information content: the more interesting or relevant the information is to daily life, the smaller k becomes (it is harder to forget). Conversely, information that is difficult to connect to existing knowledge networks, such as the meaningless syllables used by Ebbinghaus, results in a larger k (it is easier to forget).

Another concept characterising exponential decay is the *half-life*. Half-life $(T_{1/2})$ is the time required for the quantity of a substance to decrease by half, and

$$T_{1/2} = \frac{\log_e 2}{k}\,.$$

Applying this to memory allows us to calculate the time required for retention to be halved (50%). The larger the forgetting rate k, the shorter the half-life $T_{1/2}$. This concept is extremely useful for learning strategies. Understanding the half-life of a piece of knowledge allows us to pinpoint the most efficient timing for review — just before that knowledge begins to decline by more than half. By introducing the concept of half-life, we can reframe the learning challenge of "when to review" as a precise time-management problem.

Heller, Mack and Seitz [7] proposed another forgetting curve. This is interesting, because it is a combination of two exponential functions.

$$Q(t) = \mu_1 e^{-a_1} + \mu_2 e^{-a_2}\,.$$

Modern scientist are still studying the forgetting curves, and if you interested in forgetting curve, then please consult Murre and Dros [11].

5 Recovery period after exercises

The decrease (recovery) in heart rate after exercise generally follows an exponential pattern. This is because the heart rate recovery process follows an exponential decay pattern, decreasing rapidly immediately after exercise and then slowing down.

The heart rate recovery process (HRR) is typically modelled using the following equation:

$$HR(t) = (HR_{peak} - HR_{rest})e^{-t/\tau} + HR_{rest}\,,$$

where $HR(t)$ is the heart rate (beats per minute, bpm) at time t after exercise ends; HR_{peak} is the maximum heart rate immediately after exercise ends (or peak heart rate during exercise); HR_{rest} is the resting heart rate; t is the time since exercise ended (seconds or minutes) and τ is the time constant that indicates the speed of heart rate

recovery. A smaller value of τ indicates faster heart rate recovery (indicating better cardiopulmonary fitness). This model describes how the heart rate decays at a constant rate toward a target value, known as the resting heart rate HR_{rest}.

When t is small (immediately after exercise), the function declines sharply. This decline occurs because sympathetic nervous system activity, which is heightened during exercise, rapidly decreases, and parasympathetic nervous system (via the vagus nerve) activity, which lowers heart rate, is quickly reactivated.

A late gradual decline occurs: as t increases, the function asymptotically approaches HR_{rest} and the rate of decline becomes slower. Healthy individuals and athletes exhibit a curve with a small τ value (rapid recovery), showing a steep decline in the heart rate. For more details, see [3].

More detailed analyses sometimes express heart rate recovery as the sum of two exponential functions (a double exponential function). Joseph, Kanthakumar, and Tharion [8] propose the following formula:

$$y = y_0 + A_1 e^{\frac{-(x-x_0)}{\tau_1}} + A_2 e^{\frac{-(x-x_0)}{\tau_2}} .$$

We omit the explanation of the coefficients in this equation; please note that it is a double exponential function.

6 Inverse proportional function

The decreasing exponential function resembles the inverse proportional function, and in this section, we examine a phenomenon related to the inverse proportional function. The well-known Jaune's Law states that the subjective evaluation of the length of time is inversely proportional to age, meaning that the same passage of time feels longer to younger people and shorter to older people. Therefore, Jaune's Law presents the following equation.

$$\text{time} = \frac{1}{\text{age}} .$$

For example, if a year at age one is perceived as 365 days, then it feels like one-tenth of that at age 10 and one-fiftieth at age 50. There are two explanations for why Jaune's Law can be applied to human beings.

The first explanation is a physiological one. It is known that ageing slows the progression of the "mental clock", leading to discrepancies in time perception. Psychological and neuroscience experiments show that when people over 50 mentally measure two minutes, they perceive it as having passed in an average of 87 seconds, whereas those in their 20s take 115 seconds. Hence, human tends to underestimate time as they age. The primary causes include changes in neural activity and a decline in information processing speed. Furthermore, the decline in brain metabolic rate associated with ageing plays a role.

The second explanation is based on young people's frequent opportunities to process new information and continuous learning and discovery. As people age, reduced exposure to new stimuli and more efficient information processing lead to a perceived shortening of time. A decrease in new experiences affects the perception of time. As adults, life becomes patterned, and the brain processes actions as routines automatically. Therefore, older adults have simpler memories of daily events, making them feel as if they pass in the blink of an eye. A decrease in episodic memory also plays a role. Memorable events function as anchors for time; the more episodes we have, the longer time feels. When similar days continue, these memory anchors diminish, making a year feel shorter when looking back. For more information on the recent research on the relation between age and time, see [5] and [18].

7 A Japanese narrative on geometric progression

The Tale of Hideyoshi Toyotomi [13], who was an influential leader of Japan and lived from 1537 to 1598, and his subordinate Sorori Shinzaemon [12], whose job was to entertain the influential leader with telling stories or making poems. In the well-known novel and movie Shōgun, the Taikō (the deceased ruler) is a character based on Toyotomi Hideyoshi.

One day, Toyotomi Hideyoshi was impressed with Sorori Shinzaemon's performance and offered Sorori any reward he desired. Shinzaemon humbly requested a seemingly small reward. "I only ask for a single grain of rice on the first day", he said, "and on each subsequent day, you must give me double the amount of rice from the previous day. I would like this to continue for the number of squares on a shogi board, which is 81 days." Hideyoshi, thinking this was a very modest request, laughed and immediately agreed. He ordered his men to begin fulfilling the request. For the first four days, the amount of rice was negligible because Sorori got one grain, two grains, four grains, and eight grains. However, as the days passed, the amount of rice began to increase exponentially. By the middle of the period, the quantity was becoming unmanageable. On the 57th day, the total amount reaches 4.80384×10^9 tons, exceeding the predicted global grain production of 2.925×10^9 tons for 2025. The total amount of rice for the entire 81 days is a staggering number: $1 + 2 + 2^2 + 2^3 + \cdots 2^{80} = 2^{81} - 1$ grains. Suppose that 30 grains amount to 1 gram; then the total weight is $(2^{81} - 1)/(30 * 1000 * 1000) = 8.06 \times 10^{16}$ tons. Hideyoshi was astonished and realized Shinzaemon's clever request had outsmarted him. The tale serves as a lesson about the power of exponential functions and the wisdom of the ordinary person in contrast to an arrogant ruler.

8 Spread of news

Why does misinformation spread easily? Recent research[15] has revealed that inaccurate information and deliberate misinformation spread more widely and rapidly than accurate information. This phenomenon is primarily driven by two powerful factors: human emotions and novelty of information. Studies have shown that people are strongly drawn to content with high emotional value. Here, emotional value refers to the property of evoking strong emotions such as fear, anger, disgust or surprise. While calm, objective facts often struggle to capture people's interest, emotionally charged misinformation instantly grabs their attention. Specifically, misinformation that stimulates negative emotions is known to linger more strongly in people's memories than positive information and encourages sharing behaviours.

Novelty is an even more crucial factor. People are strongly drawn to surprising facts they didn't know or to unexpected developments. A large-scale study on information diffusion on Twitter pointed out that misinformation tends to be perceived as more novel than truthful. Truthful news is often perceived as less novel because it aligns with existing knowledge and common sense. In contrast, misinformation is sometimes perceived as shocking new information precisely because it seems unrealistic. This novelty becomes the driving force behind sharing rates, which can be several times higher than those for spreading truthful news. This synergy between emotion and novelty significantly increases the probability of sharing misinformation compared with sharing accurate news.

The spread of information throughout society can be modeled as follows:
(i) Initial term a: The number of people who first received misinformation.
(ii) Common ratio r: The probability that each person passes the information on to someone else.
(iii) Then, transmission proceeds as follows: The initial number of persons is a, then in the next stage, information is spread to ar persons, In the next stage, the information is spread to arr persons. In this way the total number of persons who receive information is

$$a + ar + ar^2 + ar^3 + \cdots .$$

For more details on the spread of misinformation, see [15].

9 Power Tower Mod 10

We are going to study the fastest algorithm to calculate the power tower

$$n^{n^{n^{n^{\cdot^{\cdot^{\cdot^{n}}}}}}} \mod 10 ,$$

where we use the number n for n times. This number can be expressed as $n \uparrow\uparrow n$ by using Knuth's arrow notation. It is well known that $n \uparrow\uparrow n$ can be extremely large if $n \geq 4$. When $n = 4$, this number is approximately 8.0723×10^{153}. This number

has been studied by many researchers; see [16]. As n increases, this number increases rapidly, and even the most powerful computers cannot calculate it correctly. Instead of calculating $n \uparrow\uparrow n$, many researchers have attempted to find a good algorithm for calculating $n \uparrow\uparrow n \mod m$, that is, the non-negative remainder of number $n \uparrow\uparrow n$ when divided by m, where m is an arbitrary natural number. Effective algorithms can be found in [14].

We attempted to find a good algorithm to calculate $n \uparrow\uparrow n \mod 10$, that is, the last digit of $n \uparrow\uparrow n$. We found a very curious fact, namely that $n \uparrow\uparrow n \mod 10$ depends only on $n \mod 20$. Using this fact, we can more easily calculate $n \uparrow\uparrow n \mod 10$ by calculating $n \mod 20$. This produces the fastest algorithm for calculating $n \uparrow\uparrow n \mod 10$. In our study, we denote by $\mod(p, q)$ the least non-negative remainder of the number p when dividing by q, so $\mod(p, q) = \mod(r, q)$ if and only if $p \mod q = r \mod q$.

To read this article you need to know the basic properties of congruence equations. Regarding the properties of the Carmichael functions, you do not need any knowledge other than the properties described in Lemma 5. Throughout this study, we let λ be the Carmichael function. For more knowledge about the Carmichael function, see [17]. In this article, we use the concept *relatively prime*. Two integers are relatively prime if they share no common positive factors (divisors) except for one. The results of this article were found by Kazuaki Fujii when he was a high school student. He used a computer algebra system to obtain the chart of Theorem 16. Ryohei Miyadera collaborated with Kazuaki Fujii to prove the result. They presented their research in [10] in 2003 without mathematical proof.

Lemma 5. *For non-negative integers p, q, s, t, u with $q \geq 2$,*

(i) *If* $\mod(s, q) = 1$, *then* $\mod(st, q) = \mod(t, q)$.

(ii) *If* $\mod(s, q) = 1$, *then* $\mod(s^p, q) = 1$.

(iii) *If* $\mod(s, q) = \mod(t, q)$ *and* $\mod(u, q) = \mod(v, q)$, *then* $\mod(su, q) = \mod(tv, q)$.

(iv) *If* $\mod(s, q) = \mod(t, q)$, *then* $\mod(ks, q) = \mod(kt, q)$ *for each integer* $k \geq 0$.

(v) *If* $\mod(s, 2) = \mod(t, 2)$ *and* $\mod(s, 5) = \mod(t, 5)$, *then* $\mod(s, 10) = \mod(t, 10)$.

Proof. (i) Suppose that $\mod(s, q) = 1$, and write $s = kq + 1$ and $t = hq + j$ for non-negative integers k, h, j with $0 \leq j < q$. Then $st = khq^2 + kqj + hq + j$, and hence $\mod(st, q) = j = \mod(t, q)$.

(ii) follows directly from (i).

(iii) Suppose that $\mod(s, q) = \mod(t, q)$ and $\mod(u, q) = \mod(v, q)$. Then there exist non-negative integers k, h such that $s - t = kq$ and $u - v = hq$. It therefore follows that $su - tv = (s - t)u + t(u - v) = kuq + thq$, and we obtain $\mod(su, q) = \mod(tv, q)$.

(iv) Suppose that $\mod(s, q) = \mod(t, q)$. Since $\mod(k, q) = \mod(k, q)$ for each non-negative integer k, by (iii) we obtain $\mod(ks, q) = \mod(kt, q)$.

(v) Suppose that $\mod(s, 2) = \mod(t, 2)$ and $\mod(s, 5) = \mod(t, 5)$, and let $v = \mod(s, 10)$ and $w = \mod(t, 10)$. Then $\mod(v, 2) = \mod(w, 2)$, so both v, w are odd or even, and also $\mod(v, 5) = \mod(w, 5)$, so, $v = w$ or $|v - w| = 5$ since $0 \leq v, w < 10$. Therefore, we obtain $v = w$. □

Lemma 6. [17] *Let λ be the Carmichael function. Then*

(i) $\mathrm{mod}\big(n^{\lambda(q)}, q\big) = 1$ *for any relatively positive integers* $n \geq 1$ *and* $q \geq 2$*;*
(ii) $\lambda(10) = 4$, $\lambda(5) = 4$, $\lambda(4) = 2$ *and* $\lambda(2) = 1$.

Lemma 7. *Let m and n be positive integers such that n is relatively prime to* 10*,*

$$\mathrm{mod}\big(n^{m+1}, 10\big) = \mathrm{mod}\big(n^{\mathrm{mod}(m,4)+1}, 10\big)\,. \tag{4}$$

Proof. By Lemma 6, $\mathrm{mod}\big(n^{\lambda(10)}, 10\big) = 1$ and $\lambda(10) = 4$, so $\mathrm{mod}\big(n^4, 10\big) = 1$. Divide m by 4 to find the non-negative integer p such that $m = 4p + \mathrm{mod}(m, 4)$. By (ii) of Lemma 5, we obtain $\mathrm{mod}\big(n^{4p}, 10\big) = 1$. By (i) of Lemma 5, we obtain

$$\mathrm{mod}\big(n^m, 10\big) = \mathrm{mod}\big(n^{4p+\mathrm{mod}(m,4)}, 10\big) = \mathrm{mod}\big(n^{\mathrm{mod}(m,4)}, 10\big),$$

and hence by (iv) of Lemma 5, we obtain (4). □

Lemma 8. *For any positive integers h and m, we have*

$$\mathrm{mod}\big((2^h)^{m+1}, 10\big) = \mathrm{mod}\big((2^h)^{\mathrm{mod}(m,4)+1}, 10\big)\,.$$

Proof. Since 2^h is relatively prime to 5, by Lemma 6 we obtain

$$\mathrm{mod}\big((2^h)^4, 5\big) = \mathrm{mod}\big((2^h)^{\lambda(5)}, 5\big) = 1\,.$$

There exists a non-negative integer p such that $m = 4p + \mathrm{mod}(m, 4)$. By (ii) of Lemma 5, $\mathrm{mod}\big((2^h)^{4p}, 5\big) = 1$, so by (i) of Lemma 5,

$$\mathrm{mod}\big((2^h)^m, 5\big) = \mathrm{mod}\big((2^h)^{4p+\mathrm{mod}(m,4)}, 5\big) = \mathrm{mod}\big((2^h)^{\mathrm{mod}(m,4)}, 5\big)\,.$$

Therefore by (iv) of Lemma 5, we have

$$\mathrm{mod}\big((2^h)^{m+1}, 5\big) = \mathrm{mod}\big((2^h)^{\mathrm{mod}(m,4)+1}, 5\big)\,. \tag{5}$$

It is clear that

$$\mathrm{mod}\big((2^h)^{m+1}, 2\big) = 0 = \mathrm{mod}\big((2^h)^{\mathrm{mod}(m,4)+1}, 2\big)\,, \tag{6}$$

and hence by (5), (6) and (v) of Lemma 5, the proof follows. □

Lemma 9. *For any positive integers m and n,*

$$\mathrm{mod}\big(n^{m+1}, 10\big) = \mathrm{mod}\big(n^{\mathrm{mod}(m,4)+1}, 10\big)\,.$$

Proof. If n is a multiple of 10, then $\mathrm{mod}\big(n^{m+1}, 10\big) = 0 = \mathrm{mod}\big(n^{\mathrm{mod}(m,4)+1}, 10\big)$.
If n is a odd multiple of 5, then $\mathrm{mod}\big(n^{m+1}, 10\big) = 5 = \mathrm{mod}\big(n^{\mathrm{mod}(m,4)+1}, 10\big)$.

Next we suppose that n is not a multiple of 5. Then $n = 2^h\, p$ for a non-negative integer h and a positive integer p which is relatively prime to 10. By Lemma 8,

$$\mathrm{mod}\big((2^h)^{m+1}, 10\big) = \mathrm{mod}\big((2^h)^{\mathrm{mod}(m,4)+1}, 10\big)\,,$$

so by Lemma 7, $\mathrm{mod}(p^{m+1}, 10) = \mathrm{mod}(p^{\mathrm{mod}(m,4)+1}, 10)$. Therefore by (iii) of Lemma 5,

$$\begin{aligned}\mathrm{mod}\big(n^{m+1}, 10\big) &= \mathrm{mod}\big((p \times 2^h)^{m+1}, 10\big) = \mathrm{mod}\big(p^{m+1}(2^h)^{m+1}, 10\big) \\ &= \mathrm{mod}\big((p)^{\mathrm{mod}(m,4)+1}(2^h)^{\mathrm{mod}(m,4)+1}, 10\big) = \mathrm{mod}\big(n^{\mathrm{mod}(m,4)+1}, 10\big)\,.\end{aligned}$$

□

Remark 10. By Lemma 9, $\mathrm{mod}\big(n^t, 10\big) = \mathrm{mod}\big(n^{\mathrm{mod}(t-1,4)+1}, 10\big)$ for all integers $n > 0$ and $t \geq 2$.

Lemma 11. *For any positive integer m and positive odd integer n,*

$$\mathrm{mod}\big(n^m, 4\big) = \mathrm{mod}\big(n^{\mathrm{mod}(m,2)}, 4\big)\,.$$

Proof. Since n is relatively prime to 4, by Lemma 6, we obtain

$$\mathrm{mod}\big(n^2, 4\big) = \mathrm{mod}\big(n^{\lambda(4)}, 4\big) = 1\,. \tag{7}$$

There exists non-negative integer p such that $m = 2p + \mathrm{mod}(m, 2)$. Then, by (7) and (ii) of Lemma 5, $\mathrm{mod}\big(n^{2p}, 4\big) = 1$, so by (i) of Lemma 5,

$$\mathrm{mod}\big(n^m, 4\big) = \mathrm{mod}\big(n^{2p+\mathrm{mod}(m,2)}, 4\big) = \mathrm{mod}\big(n^{\mathrm{mod}(m,2)}, 4\big)\,. \qquad \square$$

Lemma 12. *If p, q are positive integers and $\mathrm{mod}(p, q) > 0$, then $\mathrm{mod}(p-1, q) = \mathrm{mod}(p, q)-1$.*

This is a simple property of $\mathrm{mod}(,)$. We will omit the proof.

Lemma 13. *For any positive integers m and n, $\mathrm{mod}\big(n^m, 10\big) = \mathrm{mod}\big(\mathrm{mod}(n, 10)^m, 10\big)$.*

Proof. There are non-negative integers p and r such that $r = \mathrm{mod}(n, 10)$ and $n = 10p+r$. Then $\mathrm{mod}\big(n^m, 10\big) = \mathrm{mod}\big((10p + r)^m, 10\big) = \mathrm{mod}\big(r^m, 10\big)$, which finishes the proof. $\square$

Theorem 14. *For any positive odd integer n,*

$$\mathrm{mod}(n \uparrow\uparrow n, 10) = \mathrm{mod}\big(\mathrm{mod}(n, 10)^{\mathrm{mod}(n,4)}, 10\big)\,. \tag{8}$$

Proof. If $n = 1$, then (8) is trivial. We suppose that $n \geq 3$. By Remark 10, we obtain

$$\mathrm{mod}(n \uparrow\uparrow n, 10) = \mathrm{mod}\big(n^{n\uparrow\uparrow(n-1)}, 10\big) = \mathrm{mod}\big(n^{\mathrm{mod}(n\uparrow\uparrow(n-1)-1,4)+1}, 10\big)\,. \tag{9}$$

Since n is odd, $\mathrm{mod}(n \uparrow\uparrow (n-1), 4) > 0$. Hence by Lemma 12,

$$\mathrm{mod}\big(n^{\mathrm{mod}(n\uparrow\uparrow(n-1)-1,4)+1}, 10\big) = \mathrm{mod}\big(n^{\mathrm{mod}(n\uparrow\uparrow(n-1),4)}, 10\big)\,. \tag{10}$$

Since n is odd, we have $\mathrm{mod}\big(n \uparrow\uparrow (n-2), 2\big) = 1$. By Lemma 11,

$$\mathrm{mod}\big(n \uparrow\uparrow (n-1), 4\big) = \mathrm{mod}\big(n^{n\uparrow\uparrow(n-2)}, 4\big) = \mathrm{mod}\big(n^{\mathrm{mod}(n\uparrow\uparrow(n-2),2)}, 4\big) = \mathrm{mod}\big(n, 4\big)\,. \tag{11}$$

Therefore, by (9), (10), (11) and Lemma 13, we have

$$\mathrm{mod}(n \uparrow\uparrow n, 10) = \mathrm{mod}\big(n^{\mathrm{mod}(n,4)}, 10\big) = \mathrm{mod}\big(\mathrm{mod}(n, 10)^{\mathrm{mod}(n,4)}, 10\big)\,. \qquad \square$$

Theorem 15. *For any even integer $n \geq 4$,*

$$\mathrm{mod}(n \uparrow\uparrow n, 10) = \mathrm{mod}\big(n^4, 10\big) = \mathrm{mod}\big(\mathrm{mod}(n, 10)^4, 10\big)\,. \tag{12}$$

Proof. By Remark 10, we obtain

$$\mathrm{mod}\big(n \uparrow\uparrow n, 10\big) = \mathrm{mod}\big(n^{n\uparrow\uparrow(n-1)}, 10\big) = \mathrm{mod}\big(n^{\mathrm{mod}(n\uparrow\uparrow(n-1)-1,4)+1}, 10\big)\,. \tag{13}$$

Since n is even and $n \geq 4$, we have

$$\mathrm{mod}\big(n \uparrow\uparrow (n-1), 4\big) = 0\,, \tag{14}$$

and hence we have

$$\mathrm{mod}(n \uparrow\uparrow (n-1) - 1, 4) + 1 = 4\,. \tag{15}$$

Therefore, by (13), (14), (15) and Lemma 13, we have (12). □

Theorem 16. *For any positive integer $n \neq 2$,*

$$\mathrm{mod}(n \uparrow\uparrow n, 10)$$

depends only on $\mathrm{mod}(n, 20)$.

Proof. By Theorems 14 and 15 for any positive integer $n \neq 2$, the value of

$$\mathrm{mod}(n \uparrow\uparrow n, 10)$$

depends only on $\mathrm{mod}(n, 10)$ and $\mathrm{mod}(n, 4)$, i.e., on $\mathrm{mod}(n, 20)$. □

We use Theorems 14 and 15 to obtain the following values of $\mathrm{mod}(n \uparrow\uparrow n, 10)$.

$\mathrm{mod}(n, 20)$	$\mathrm{mod}(n \uparrow\uparrow n, 10)$
0	0
1	1
2	$6(n > 2)\,, 4(n = 2)$
3	7
4	6
5	5
6	6
7	3
8	6
9	9

$\mathrm{mod}(n, 20)$	$\mathrm{mod}(n \uparrow\uparrow n, 10)$
10	0
11	1
12	6
13	3
14	6
15	5
16	6
17	7
18	6
19	9

Example 17. Suppose, for instance, that you want to calculate $\mathrm{mod}(27 \uparrow\uparrow 27, 10)$. Since $\mathrm{mod}(27, 20) = 7$, by the chart above, we obtain $\mathrm{mod}(27 \uparrow\uparrow 27, 10) = 3$.

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Appendix

Code A.

```
(*Parameter Settings*)r=0.1;
K=100;
P0=10; (*Initial Population*)
timeLimit=100;
(*Solve the Logistic Differential Equation using NDSolve*)
solution=NDSolve[{P'[t]==r*P[t]*(1-P[t]/K),P[0]==
P0},P,{t,0,timeLimit}];

(*Plotting the Result*)
Plot[P[t]/. solution,{t,0,timeLimit},PlotLabel->
"Continuous-Time Logistic Equation (K=100, r=0.1)",AxesLabel->
{"Time (t)","Population (P)"},PlotRange->All]
```

Code B.

```
Clear[R0,I0,s,i];R0=3;
GAMMA=1/14;
POPULATION=100000;
BETA=R0*GAMMA/POPULATION;
I0=1;
{susceptible,infectious,recovered}=NDSolveValue[{s'[t]==
-BETA*s[t]*i[t],i'[t]==BETA*s[t]*i[t]-GAMMA*i[t],r'[t]==
GAMMA*i[t],s[0]==POPULATION-I0,i[0]==I0,r[0]==0},{s,i,r},{t,0,150}]
Plot[{susceptible[t],infectious[t],recovered[t]},{t,0,150},
PlotStyle->{Dashing[Large],Dashing[Tiny],Dashing[Medium]},
PlotLegends->{"Susceptible","Infectious","Removed"}]
```

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