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Starter packs for geometric constructions

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1 Introduction

An interesting part of the history of geometry is the gradual refinement of the "starter pack" sufficient to perform the standard geometric constructions. There are two parts to such a starter pack: the geometric tools [1, 6, 8, 7, 15, 16, 12, 17, 19]; and the initial data set of points (points, lines, circles, squares, etc.) [18, pp.104–117]. For instance, Euclid's [4] starter pack involved a straightedge and compass together with two given points. Mohr [10] and Mascheroni [9] removed the straightedge from Euclid's pack, showing that all standard geometric constructions which are achievable with a compass and straightedge could, in fact, be carried out using a compass alone. Poncelet [11] and Steiner [14], on the other hand, removed the compass from Euclid's pack, under the assumption that the initial data set was a single circle together with its center, and showed how this enabled standard geometric constructions with a straightedge only.

In contrast, Hilbert [5, p.152] demonstrated that a starter pack consisting of only a straightedge and a given circle *without* its centre point was insufficient for the standard geometric constructions (since it is impossible to construct the centre point of the given circle).

Severi [13] kept the straightedge and made the starter pack of Poncelet and Steiner even smaller, by showing that their initial data set of circle with centre point could be replaced by an arc (of any length) together with its centre point.

Cauer [3] varied the above starter packs once again. Although the centre of a given circle is not constructible using only a straightedge, Cauer showed that the centres of *two* given intersecting circles may nevertheless be recovered with the use of a straightedge alone.

I have summarised the evolution of geometric construction starter packs in Table 1.

The above progression naturally raises the following questions:

Can Cauer's starter pack be refined? How so?

These questions were examined in the recent coursework Master's thesis of Yuyan Cai [2]. Allow me to illustrate and summarise one of the main results from Cai's thesis: that two full circles in Cauer's starting pack can be replaced by two intersecting arcs, in the sense that the centers can still be constructed using only a straightedge. Therein, Cai proposed the starter pack for constructive geometry listed in the bottom row of Table 1.

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Geometer	Starter Pack Tools	Starter Pack Points
Euclid (~ 300 BCE)	Straightedge + compass	Two distinct points
Mohr (1672)	Compass alone	Two distinct points / unit segment
Mascheroni (1797)	Compass alone	Two distinct points / unit segment
Poncelet–Steiner (1822–1833)	Straightedge alone	One circle with known center
Hilbert (1891–1895)	Straightedge alone	A circle with unknown center (insufficient)
Severi (1904)	Straightedge alone	A circular arc with known center
Cauer (1912)	Straightedge alone	Two intersecting circles with unknown centres
Cai (2024)	Straightedge alone	Two intersecting arcs with unknown centres

Table 1: Evolution of “starter packs” in constructive geometry.

2 Revisiting Cauer’s proof

An example of Cauer’s initial data set is seen in Figure 1. By revisiting Cauer’s proof, we will see that the minor arcs in the initial data set of the two intersecting circles play no role in constructing the centre points with only a straightedge, and so they can be removed to leave a shape of two intersecting major arcs; see Figure 2.

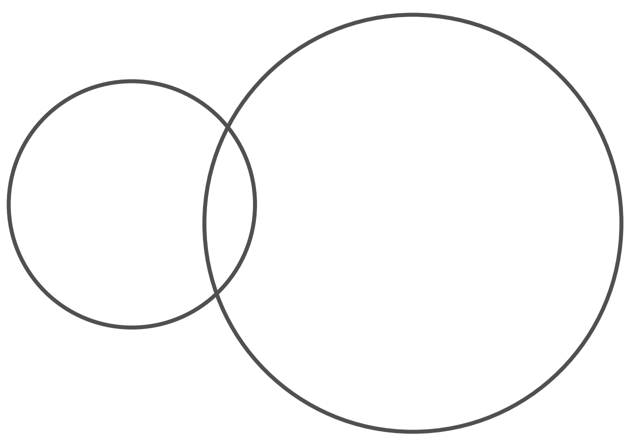

Figure 1: Cauer’s initial data set.

So, rather than a starter pack consisting of a straightedge and Figure 1, we remove the minor arcs of both circles to form a starter pack of straightedge and two intersecting major arcs with unknown centres.

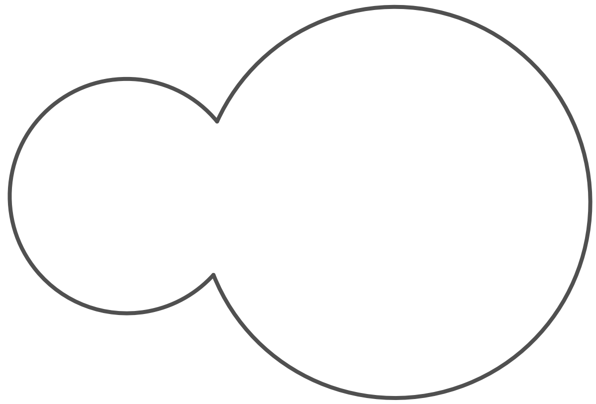

Figure 2: New initial data set.

Observe that the two major arcs lie on opposite sides of the chord joining their two points of intersection in Figure 2. Geometrically, that gives a "double-lobed" shape: one bulge on each side.

In order to construct the centre of one of the given circles with only a straightedge, Cauer's approach was to choose two points A and B on one of the circles and then form four extended segments from A and B through the circles' intersection points C and D, giving rise to the points E, F, G and H on the other circle; see Figure 3.

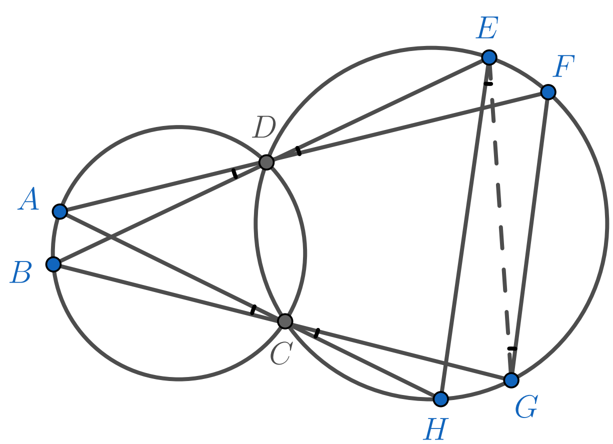

Figure 3: Cauer's construction.

According to the properties of circles, angles that subtend the same arc are equal. Thus,

$$\angle ADB = \angle BCA\,, \quad \angle EDF = \angle EGF\,, \quad \angle HEG = \angle HCG\,.$$

Now, $\angle ADB$ and $\angle EDF$ are opposite angles; thus,

$$\angle ADB = \angle EDF.$$

And, we can prove $\angle ADB = \angle EGF$. Similarly,

$$\angle BCA = \angle HEG.$$

We thus obtain $\angle HEG = \angle EGF$ and conclude that the line segment EH is parallel to the line segment GF. Thus, the quadrilateral $EFGH$ is a trapezium whose bases are EH and FG.

Extend the sides EF and HG until they intersect at a point P. Next, construct the diagonals EG and FH, and let their intersection point be Q. By Steiner's theorem for trapeziums [14], the following four points are collinear:

the midpoint of EH,

the midpoint of FG,

the intersection point $P = EF \cap HG$,

the intersection point $Q = EG \cap FH$.

Hence, the line PQ passes through the midpoints of the parallel chords EH and FG.

Since the perpendicular bisector of a chord passes through the centre of a circle, and parallel chords share a common perpendicular through the centre, it follows that the line PQ contains the centre of the circle.

If we repeat the above process using a different choice of points A and B on the same circle, then we obtain a second line containing the centre of the circle. Since the new line is distinct from PQ, their intersection determines the unique centre point.

By scrutinising Cauer's approach, Yuyan Cai [2] discovered that the minor arcs of both circles played no role in the construction. Thus, it is possible to re-run the construction without the minor arcs; see Figure 4.

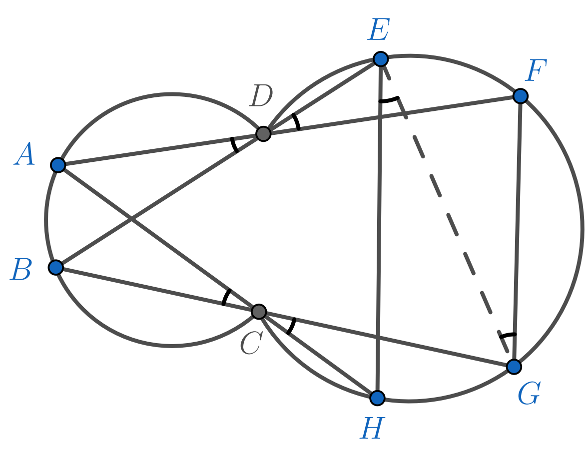

Figure 4: New construction.

3 Conclusion

The history of geometric constructions reveals a gradual simplification and alteration of the "starter pack" of Euclid. In this article, an interesting result due to Yuyan Cai [2] was presented. Cai revisited Cauer's construction for two intersecting circles and observed that the minor arcs of the circles played no role in the straightedge construction of the centre points. By removing these arcs a smaller starter pack consisting only of a straightedge together with two intersecting major arcs was obtained. Nevertheless, the centres of the underlying circles could still be constructed.

Thus, somewhat surprisingly, the full circles in Cauer's starter pack may be replaced by intersecting arcs without losing the ability to recover the centres. This provides another example of how very small amounts of geometric information can still encode substantial Euclidean structure.

It would be interesting to determine how far this reduction process can continue. For example, what other partial-circle configurations still contain enough information to recover the centres constructively?

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