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Applications of permutation graphs to problem-solving

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1 Introduction

The famous Scottish magician Alex Elmsley often began a trick with a deck of ordered cards and then shuffled it multiple times. To the audience's amazement, the deck would *magically* be returned to its original configuration. Would it surprise you to hear that the trick doesn't involve magic, but actually maths? The underlying concept behind it is permutation graphs. This paper will teach you how to recreate this trick and solve other fun puzzles.

Permutation graphs were first introduced and discussed in 1967 by Chartrand and Harary in their paper *Planar Permutation Graphs* [1]. Over the last few decades, several applications of permutation graphs have been developed in data sorting, comparative genomics and network analysis. This paper starts by describing permutations and graphs as independent concepts.

A permutation is a way that a set can be ordered or arranged. For example, the ordered set $(1, 2, 3, 4, 5, 6)$ can be permuted to $(3, 2, 6, 5, 4, 1)$. While the elements in both sets are the same, the order has been changed. The number of permutations of the set $(1, 2, 3, 4, 5, 6)$ is $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6!$ because any of the 6 numbers can go into the first position, any of the remaining 5 numbers can go into the second position, and so on.

Graphs are defined as mathematical structures that are used to model pairwise relationships between objects. A graph consists of a set of vertices (or nodes) and a set of edges that connect some pairs of vertices. A formal definition of a graph G is $G = (V, E)$, where V is the set of vertices and E is the set of edges. For example, Figure 1 illustrates the graph $G = (\{1, 2, 3, 4\}, \{12, 13, 23, 24, 34\})$.

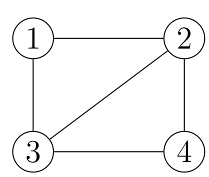

Figure 1: A graph G

There are two types of graphs: directed and undirected graphs. In a directed graph, each edge has a direction, and is represented by an arrowhead indicating directionality between vertices. In an undirected graph, each edge does not have a direction, so no

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arrowheads are used in the edges. Figure 2 illustrates the directionality difference between undirected and directed graphs.

Figure 2: Undirected graph (left) vs directed graph (right)

Permutation graphs combine the two concepts of permutations and graphs. They are directed graphs, where the vertices represent the elements of a permutation. In the original example where $(1, 2, 3, 4, 5, 6)$ is permuted to $(3, 2, 6, 5, 4, 1)$, this means that element 1 is being directed to position 3; element 2 is being directed to position 2; element 3 is being directed to position 6; element 4 is being directed to position 5; element 5 is being directed to position 4; and element 6 is being directed to position 1.

Another way to think about permutation graphs is that the first set of numbers can be viewed as inputs to a program or a function, and the second set of numbers represents the output, i.e., where the inputs are being directed. Call the inputs i and the outputs π_i. This is illustrated in Figure 3.

$$
\begin{array}{ccc}
i & & \pi_i \\
1 & \longrightarrow & 3 \\
2 & \longrightarrow & 2 \\
3 & \longrightarrow & 6 \\
4 & \longrightarrow & 5 \\
5 & \longrightarrow & 4 \\
6 & \longrightarrow & 1
\end{array}
$$

Figure 3: Directed permutation from i to π_i

A permutation graph can also be visualized as a collection of directed cycles. A cycle is a sequence of elements that are directed in a closed loop. The first element is directed to the second element; the second element is directed to the third element; and so on, until the last element is directed to the first element. For example, in Figure 4, $(1, 3, 6)$ is a cycle; (2) is a cycle; and $(4, 5)$ is a cycle. The number of cycles in a permutation graph is the total number of such loops. Therefore, the permutation graph in Figure 4 has 3 cycles.

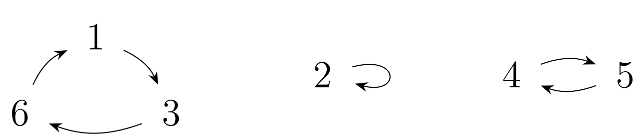

Figure 4: Directed permutation graph

2 Permutation graph properties and theorems

Certain properties of permutation graphs are defined, and two theorems that are very useful in problem-solving applications are shown below.

The indegree and outdegree are important properties of the vertices of any directed graph. The *indegree* of a vertex is the number of edges leading toward the vertex, and the *outdegree* of a vertex is the number of edges leading away from the vertex. Therefore, all vertices of a permutation graph have indegree 1 and outdegree 1.

If n is the number of vertices in the permutation graph, and c is the number of cycles, then the *sign* of the permutation is defined as $(-1)^{n-c}$, giving the sign a value of either $+1$ or -1. For example, the permutation graph $(3, 2, 6, 5, 4, 1)$, shown above, has $n = 6$ elements and $c = 3$ cycles. Therefore, the sign is $(-1)^{6-3} = (-1)^3 = -1$. This paper will discuss a more conceptual way of understanding the sign of a permutation graph after Theorem 2.

Theorem 1. *A permutation graph is a union of disjoint directed cycles.*

Proof. Consider a maximal sequence of numbers, each directed at the next, that has no repeated elements. Without loss of generality, this sequence starts with 1, then goes to 2 and 3, and so on until getting to x. This is illustrated in Figure 5.

$$\begin{tikzcd}
& 1 \arrow[r] & 2 \arrow[dr] & \\
x & & & 3 \arrow[dl] \\
& \cdots \arrow[ul] & 4 \arrow[l] &
\end{tikzcd}$$

Figure 5: Disconnected cycle

From x, you have to be directed to another numbers, say y. Since the sequence is maximal, y must be one of the numbers $1, \ldots, x$. Assume that y is not 1. Figure 6 illustrates this, where, without loss of generality, $y = 3$.

$$\begin{tikzcd}
& 1 \arrow[r] & 2 \arrow[dr] & \\
x \arrow[rrr] & & & 3 \arrow[dl] \\
& \cdots \arrow[ul] & 4 \arrow[l] &
\end{tikzcd}$$

Figure 6: Directed path with a backwards edge

In Figure 6, element 3 now has an indegree of 2. Therefore, this cannot be a permutation graph, which is a contradiction. Thus, for a permutation graph, x cannot connect to any earlier element in the chain other than $y = 1$. Hence, we have a cycle. Now, consider some element in the graph that is not in this cycle, and repeat the above argument to find another cycle, and so forth. It follows that a permutation graph is a collection of disjoint directed cycles. □

Theorem 2. *Consider any two vertices of a permutation graph. If the vertices that their outgoing edges point to are swapped, then the sign of the permutation graph changes.*

Proof. Evaluate the swaps in two separate cases.

Case 1: The two elements being swapped are in the same cycle.

This results in the cycle becoming two cycles. For instance, swapping 2 and k from the cycle shown in Figure 7, so that 1 now points to k instead of 2, and $k-1$ now points to 2 instead of k, results in the two cycles shown in Figure 8. The number of elements

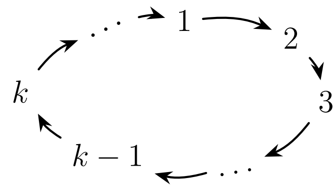

Figure 7: Original single cycle

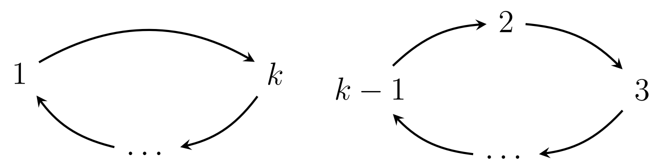

Figure 8: New cycles

is the same but the number of cycles has been increased by 1. This will decrease the parity $n-c$ of the original graph to $n-(c+1)=(n-c)-1$, a decrease by 1, and the sign of the graph changes.

Case 2: The two elements being swapped are in two different cycles. This replaces the two cycles by one cycle. For instance, swapping 2 and k from the cycles shown in Figure 9, so that 1 now points to k instead of 2, and $k-1$ now points to 2 instead of k, results in one fewer cycle, as shown in Figure 10.

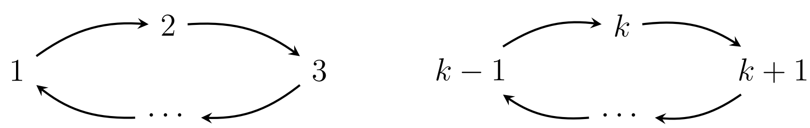

Figure 9: Original split cycle

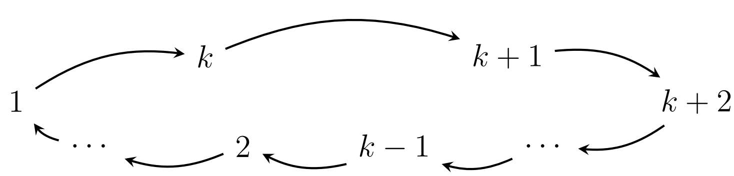

Figure 10: New single cycle

Now calculate the parity of the new sequence again. The number of elements n remains constant but c has decreased by 1. This changes the parity of $n-c$, which changes the sign of the new graph. □

If we start with a permutation graph G with n vertices and c cycles, then by Theorem 2 and its proof, we can keep using swaps to break cycles into smaller cycles, until we reach the *identity graph* which just consists of n vertices and n cycles, namely a loop at each vertex. The number of swaps used is $n-c$. If this number is even, then G has sign 1 and is called an *even* permutation. On the other hand, if $n-c$ is odd, then G has sign -1 and is called an *odd* permutation.

3 Applications to problem-solving

Permutation graphs are a useful tool when solving counting problems. They are especially powerful when encountering problems that require proof of whether a certain pattern or outcome is achievable. This paper will look at examples of problems where permutation graphs can be used to come up with elegant solutions.

Problem 1. *Find all values of n for which the set $(1, 2, \ldots, n-1, n)$ can be transformed to $(n, n-1, \ldots, 2, 1)$ by an even number of element swaps.*

Solution: Consider the cases corresponding to remainders when n is divided by 4.

Case 1: $n \in \{4k, 4k+1\}$.
In this case, Figures 11 and 12 below show how to make $2k$ swaps - an even number - to transform $(1, 2, \ldots, n-1, n)$ into $(n, n-1, \ldots, 2, 1)$.

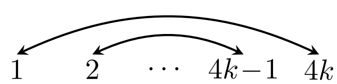

Figure 11: Swaps for when $n = 4k$

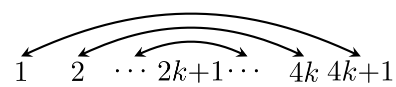

Figure 12: Swaps for when $n = 4k+1$

Case 2: $n = 4k+2$.
The permutation graph of the original sequence $(1, 2, \ldots, n)$ is the identity graph with n vertices and $c = n$ (loop) cycles, since every element in the sequence is mapped to itself. Consequently, its sign is $(-1)^{n-c} = (-1)^0 = 1$.

By Theorem 2, every swap changes the sign, so, after an even number of swaps, the sign is still 1. In the sequence $(4k+2, 4k+1, \ldots, 2, 1)$, there are $2k+1$ cycles, as shown in Figure 13. The sign of the graph is therefore $(-1)^{4k+2-(2k+1)} = (-1)^{2k+1} = -1$, so it is impossible to go from the original sequence to its reverse by an even number of swaps.

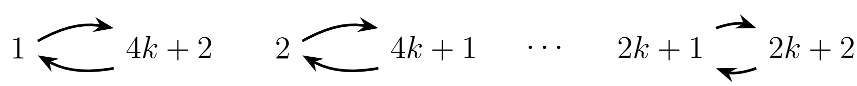

Figure 13: Permutation graph when $n = 4k+2$

Case 3: $n = 4k+3$.
As in Case 2, the original sequence has sign 1 but the sequence $(4k+3, 4k+2, \ldots, 2, 1)$ has sign $(-1)^{n-c} = (-1)^{4k+3-(2k+2)} = -1$, since it has $2k+2$ cycles, as shown in Figure 14. Again by Theorem 2, it is not possible to make an even number of swaps to transform the original sequence into its reverse sequence.

Figure 14: Case $n = 4k+3$

Therefore, n must be of the form $4k$ or $4k+1$ to change $(1, 2, \ldots, n-1, n)$ into $(n, n-1, \ldots, 2, 1)$ by an even number of element swaps. □

Now, review how this concept can be applied to a more advanced problem where it is not immediately clear how permutation graphs can be helpful. It may be tricky to find the idea immediately, so take some time to think about this next problem before looking at the solution.

Problem 2. Consider a stack of five blocks in the following configuration:

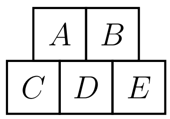

Figure 15: Starting configuration

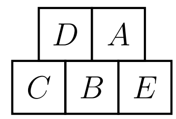

Figure 16: New looped arrangement

Suppose that the blocks are rearranged according to the rule that, if the three blocks x, y and z all touch each other, then block x can move to the position of block y; block y can move to the position of block z; and block z can move to the position of block x. This kind of rearrangement is called a *loop*. For example, in the starting configuration in Figure 15, it is possible to loop A, B and D to make the configuration in Figure 16.

Is it possible to obtain Figure 18 from Figure 17?

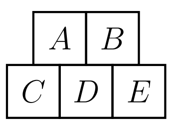

Figure 17: Initial arrangement

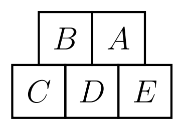

Figure 18: Desired arrangement

Solution: Let us call A, B, C, D and E as $1, 2, 3, 4, 5$, respectively, as it can aid us in using permutation graphs, and let us label the original graph as $(1, 2, 3, 4, 5)$.

Let a state in the graph be $(\pi_1, \pi_2, \pi_3, \pi_4, \pi_5)$. From this state, there are three possible moves, given by the three cases below.

Case 1: Loop π_1, π_3, π_4 to make the graph $(\pi_3, \pi_2, \pi_4, \pi_1, \pi_5)$; see Figure 19.

Case 2: Loop π_1, π_2, π_4 to make the graph $(\pi_4, \pi_1, \pi_3, \pi_2, \pi_5)$; see Figure 20.

Case 3: Loop π_2, π_4, π_5 to make the graph $(\pi_1, \pi_4, \pi_3, \pi_5, \pi_2)$; see Figure 21.

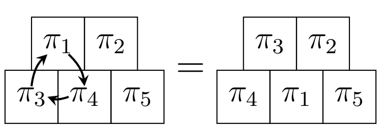

Figure 19: Loop π_1, π_3, π_4

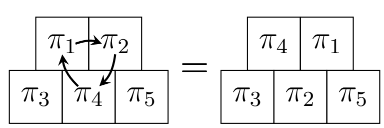

Figure 20: Loop π_1, π_2, π_4

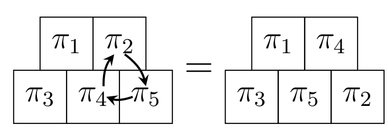

Figure 21: Loop π_2, π_4, π_5

These cases can also be obtained by swapping elements twice:

Case 1: Swap π_1, π_3 and then swap π_1, π_4.

Case 2: Swap π_1, π_2 and then swap π_2, π_4.

Case 3: Swap π_2, π_4 and then swap π_2, π_5.

Since each of the three swaps can be performed by swapping two pairs of elements, the sign of the permutation will remain the same after every rearrangement.

The original permutation graph $(1, 2, 3, 4, 5)$ has 5 elements and 5 loops, and thus has sign 1. The end state is $(2, 1, 3, 4, 5)$ which has 4 cycles, and its sign is $(-1)^{5-4} = -1$. However, as seen above, the final formation has a sign that is the opposite of the sign of the original formation. Therefore, it is impossible to get the final position. $\square$

This problem can also be solved by using a computer program to brute-force find all possible solutions. For those who want to try and code it themselves, make three functions: one that turns the left three cubes, one that turns the middle three cubes, and one that turns the last three cubes. Label every possible arrangement as a string of digits, and make a list to keep track of all combinations. Go through all new potential cases for these strings when they are put into the functions created. Check to see whether the intended string "BACDE" has been made.

Problem 3 (The 15 Puzzle)**.** The 15 Puzzle is a famous game in which the tiles of a 4×4 grid are numbered 1 to 15, with one empty square. A move is made by sliding a numbered tile into the empty square, whereby the empty square is now where the numbered tile was. For example, sliding the tile with 3 into the blank square in left-hand grid of Figure 22 gives the new arrangement in the right-hand side of Figure 22.

13	12	8	1
11	5	6	9
	10	2	6
3	4	14	7

$\longrightarrow$

13	12	8	1
11	5	6	9
3	10	2	6
	4	14	7

Figure 22: Change in position after one move

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Figure 23: Desired end-state

The objective of the puzzle is to arrange the tiles in order from 1 to 15 with the blank space in the bottom right corner as shown in Figure 23.

Is it possible to start with the position in Figure 24 and reach the desired end-state? If not, then prove why not. What about the starting position in Figure 25?

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

Figure 24

3	7	15	2
11	1		8
5	10	13	6
4	9	12	14

Figure 25

Solution: First consider the starting position in Figure 24, and define the grid as a permutation graph where the ordered set is the numbers from left to right, top to bottom, excluding the blank. For example, the left side of Figure 22 would be represented as $(13, 12, 8, 1, 11, 5, 6, 9, 10, 2, 6, 3, 4, 14, 7)$, and the right side would be represented as $(13, 12, 8, 1, 11, 5, 6, 9, 3, 10, 2, 6, 4, 14, 7)$.

All moves can be categorised into one of two types: a move where a tile is shifted horizontally, and a move where a tile is shifted vertically. When sliding a tile horizontally, the permutation graph doesn't change. For example, in the left-hand side of Figure 22, sliding tile 8 to the blank, the permutation graph would stay the same. The other case is moving a tile vertically. Consider the permutation in Figure 22. Observe how tile 3 moved left three spaces. If, instead, tile 15 had moved down to the blank, then 11 would have moved to the right three spaces. Therefore, every vertical move of a tile shifts the tile number by three spaces either left or right in the permutation graph.

Convert the above movements into swaps of the permutation graph. In a horizontal tile move, the permutation graph stays the same, so there are zero swaps. The vertical tile move shown in Figure 22 is equivalent to first swapping 3 with 6; then swapping the new position of 3 with 2; and finally, swapping the new position of 3 with 10. Therefore, moving a tile vertically is the same as swapping three elements of the permutation graph. Observe how with every swap in a permutation graph, its sign changes. Thus, any horizontal move has no sign change, and any vertical legal move has its sign change ($(-1)^3 = -1$) and the blank's row moves up or down by one.

The desired end state has 15 elements, and every element cycles back to itself, resulting in 15 cycles. Therefore, the sign of the permutation graph is $(-1)^{15-15} = 1$, with the blank space on the fourth row.

Now determine the sign of the permutation graph in Figure 24. The permutation graph for this grid can be written as $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 14)$. The number of cycles is $c = 14$: elements 1 to 13 all cycle to themselves, while 14 and 15 cycle to each other. Therefore, the sign of the permutation graph is $(-1)^{15-14} = -1$, and the blank space is on the fourth row.

Since every vertical move changes the sign of the graph, an odd number of vertical moves is needed. However, with an odd number of vertical moves, it would not be possible for the blank to end up on a row with the same parity as it started (the fourth row). Hence, it is impossible to reach the desired end-state from this starting position.

Now consider the starting position in Figure 25. The permutation graph for this grid is $(3, 7, 15, 2, 11, 1, 8, 5, 10, 13, 6, 4, 9, 12, 14)$ which has the two cycles in Figure 26. The first is from 1 to 3 to 15 to 14 to 12 to 4 to 2 to 7 to 8 to 5 to 11 to 6 and back to 1. The other is 9 to 10 to 13 and back to 9. The sign of the graph is $(-1)^{n-c} = (-1)^{15-2} = -1$.

Therefore, an odd number of vertical moves is needed because the sign needs to change from -1 to 1. However, with an odd number of vertical moves, it would not be possible for the blank to end up on a row with the same parity as the row in which it started (row 2 to row 4). Therefore, it is impossible to reach the final desired state from this starting position. $\square$

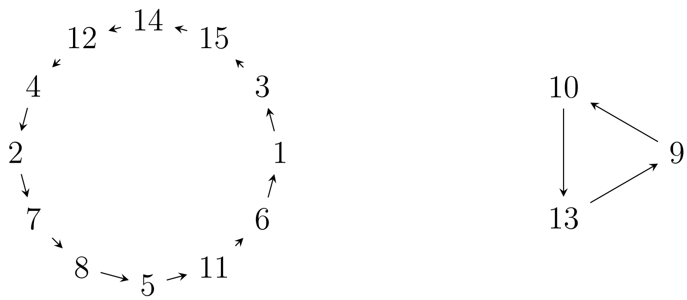

Figure 26: Graphical representation of Figure 25

4 Derangements

Derangements are a special type of permutation graph. A *derangement* is a permutation of the elements of a set in which no element appears in its original position. For example, a derangement of $(1, 2, 3)$ is $(2, 3, 1)$ but not $(3, 2, 1)$ because 2 is in its original position. In terms of permutation graphs, a derangement is a permutation graph with no loop. For example, the permutation in Figure 27 is not a derangement since it has a loop (from 2 to itself).

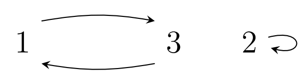

Figure 27: A non-derangement

Problem 4. *Calculate the number of derangements on n elements.*

Solution: Let $D(n)$ represent the number of derangements on n elements. Calculate $D(n)$ by performing casework on the cycle length passing through element 1.

First, look at the number of derangements where this length is 2. The element connected to 1 can be chosen in $\binom{n-1}{1}$ ways and arranged in $1!$ ways. The remaining $n-2$ elements can be arranged in $D(n-2)$ ways. Therefore, the total number of ways with element 1 in a cycle of length 2 is $\binom{n-1}{1}1!D(n-2)$. Now, looking at the number of ways in which the length of the cycle going through 1 is k, this number is $\binom{n-1}{k}k!D(n-1-k)$. Summing up all of the lengths gives

$$D(n) = \sum_{k=1}^{n-1} \frac{(n-1)!}{(n-1-k)!} D(n-1-k)\,,$$

so

$$\begin{aligned} D(n) - (n-1)D(n-2) &= \sum_{k=2}^{n-1} \frac{(n-1)!}{(n-1-k)!} D(n-1-k) \\ &= \sum_{k=1}^{n-2} \frac{(n-1)!}{(n-2-k)!} D(n-2-k) \end{aligned}$$

$$
\begin{aligned}
&= (n-1)\sum_{k=1}^{n-2} \frac{(n-2)!}{(n-2-k)!}\, D(n-2-k) \\
&= (n-1)D(n-1)\,.
\end{aligned}
$$

This gives us the following recursive formula for calculating $D(n)$ for any n:

$$
D(n) = (n-1)\big(D(n-1) + D(n-2)\big)\,. \tag{1}
$$

Let's now apply this formula to a solve a problem.

Problem 5. 10 students take a maths test. The teacher hands each test to a student to grade. Obviously, the teacher can't give someone their own test to grade since they can't trust the score that the student will give themselves. In how many different ways can the teacher hand back the tests to the students?

Solution: First, let us calculate $D(1)$. With just one student, the test must be given back to themselves, but this is not allowed, leaving us with $D(1) = 0$. If there are two students, then they have to receive each other's tests, so $D(2) = 1$. For the rest of the values, use the recursive formula (1):

$$
\begin{aligned}
D(3) &= (3-1)\big(D(2) + D(1)\big) = 2 \times 1 = 2 \\
D(4) &= (4-1)\big(D(3) + D(2)\big) = 3 \times 3 = 9 \\
D(5) &= (5-1)\big(D(4) + D(3)\big) = 4 \times 11 = 44 \\
D(6) &= (6-1)\big(D(5) + D(4)\big) = 5 \times 53 = 265 \\
D(7) &= (7-1)\big(D(6) + D(5)\big) = 6 \times 309 = 1{,}854 \\
D(8) &= (8-1)\big(D(7) + D(6)\big) = 7 \times 2119 = 14{,}833 \\
D(9) &= (9-1)\big(D(8) + D(7)\big) = 8 \times 16687 = 133{,}496 \\
D(10) &= (10-1)\big(D(9) + D(8)\big) = 9 \times 148329 = 1{,}334{,}961\,.
\end{aligned}
$$

The number of ways in which the teacher can hand back tests to grade is $1{,}334{,}961$. $\square$

5 Practice problems

Here are a few practice problems to help improve understanding of permutation graphs.

Problem 6. Take a closer look at Alex Elmsley's magic trick which was briefly discussed at the start of the paper. Let's unveil the curtain and see the math behind this seemingly impossible trick. To simplify the problem, consider just 16 cards; the same logic holds for a full deck of cards. Number the cards from 1 to 16. A *perfect out-shuffle* divides the deck into equal halves, and the cards are then alternatively interlaced, starting with the top half.

(a) If these perfect out-shuffles are repeated, then can card 4 ever go to the 10th position? Can card 8 ever go to the 11th position?

(b) Will the cards ever get back to their original positions after some number of perfect out-shuffles? If yes, then what is the minimum number of perfect out-shuffles needed for each card to return to its original position?

Solution:
(a) The first shuffle turns $(1, 2, \ldots, 16)$ into $(1, 9, 2, 10, 3, 11, 4, 12, 5, 13, 6, 14, 7, 15, 8, 16)$. In other words, $\pi_1 = 1$, $\pi_{16} = 16$ and $\pi_i \equiv 2i - 1 \pmod{15}$ for $2 \leq i \leq 15$. Note that when $2i - 1 \equiv 0 \pmod{15}$, the element is directed to 15, not 0, because no element 0 has been defined.

Now use this to track the progression of each card:

$$\begin{aligned}
1 &\to 1\\
2 &\to 3 \to 5 \to 9 \to 2\\
4 &\to 7 \to 13 \to 10 \to 4\\
6 &\to 11 \to 6\\
8 &\to 15 \to 14 \to 12 \to 8\\
16 &\to 16
\end{aligned}$$

Item i can go to position j only if i and j are in the same cycle. Therefore, card 4 will go to position 10 after three perfect out-shuffles, and card 8 can never go to position 11.

(b) The six cycle lengths above are $1, 4, 4, 2, 4, 1$. The cards will all go back to their original position after $\text{lcm}(1, 4, 4, 2, 4, 1) = 4$ out-shuffles. □

Alex Elmsley's trick: A 52-card deck returns to its original position after 8 out-shuffles. It is up to the reader to prove this. (Hint: when the cycles are written out, there will be six cycles of length 8, one cycle of length 2 and two cycles of length 1.)

Problem 7. The 100 *Prisoner Problem* [2, 3] is a problem with a famously counter-intuitive answer. A warden has 100 prisoners on death row, who are assigned numbers from 1 to 100. He offers them a last chance at survival. A room in the prison contains 100 drawers, and each drawer contains a unique prisoner's number. The prisoners will enter the room one after the other and will each open 50 of the 100 drawers. If all prisoners find their number in one of these 50 searches, then they all go free, but if even one of the prisoners doesn't find their number, they are all executed. The prisoners have time to discuss a strategy with each other before entering the room, but any prisoner who has visited the room can't give information to any other prisoner.

(a) How can you use permutation graphs to come up with a strategy that greatly enhances the probability of survival?

(b) If you are an outside observer and can swap two drawers, then can you guarantee that all prisoners will escape? If not, why not? If so, how?

Solution: (sketch)
(a) You may think that each prisoner has a $\frac{1}{2}$ chance of getting their number, so the probability that all 100 prisoners get their number is $(\frac{1}{2})^{100}$. This is approximately $10^{-28}\%$, which is essentially a zero chance of survival.

This is one of the most interesting applications of permutation graphs. What if you were told that you can improve the prisoners' odds of survival to almost $1/3$ using permutation graphs – wouldn't that be amazing! Milan Pahor's paper [3] on the 100

Prisoner Problem is an excellent read and goes through this problem in detail. You are strongly encouraged to read it.

(b) You need to ensure that the longest cycle is at most 50. You can also look at the proof of Theorem 2 and see what happens when you swap two elements. $\square$

Problem 8. Here are some problems for analysing permutation graphs by computer.

(a) Write a program to find the longest cycle length of any given permutation graph.

(b) Write a program to find the sign of any given permutation graph.

(c) Write a program to find the number of derangements for any n.

Solution: To solve (a) and (b), choose any element and keep applying the permutation until you end up back at the element. This gives the cycle containing that element, and therefore also the cycle length. Choose any element outside of the cycle and find the cycle that it is part of, and continue until all cycles have been found. This gives you the length of the longest cycle, solving (a), and it gives you the number of cycles, from which you can find the sign of the permutation graph, solving (a).

(c) Use the recursive formula $D(n) = (n-1)(D(n-1)+D(n-2))$ in (1), and the starting values $D(1) = 0$ and $D(2) = 1$, for instance using recursion or just an array. $\square$

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[3] M. Pahor, The 100 Prisoner Problem, *Parabola* **60(2)** (2024).

[4] Stanford University Mathematics Camp (SUMaC), 2022 Admissions Exam: Question 4.