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# Using partial differential equations to model the economic productivity and total monetary output of a region

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## 1 Introduction

Economic modelling involves understanding how productivity changes over space and time. One notable phenomenon in economics is agglomeration, a concept first studied by Alfred Marshall [1] in 1890. He observed that regions of high productivity tend to not only increase their own output but also boost that of surrounding areas. This is akin to a feedback loop of economic success, where thriving businesses attract more businesses, skilled workers, and infrastructure, creating a self-reinforcing cycle of growth.

In this article, we formulate three mathematical models of increasing complexity to describe how the economic productivity and total monetary output (current value of all goods and services associated) of a region evolves over time. The analysis is based on the works of Marshall [1] and Verhulst [2]. We start by assuming unlimited productivity growth, then adding constraints using productivity to find total monetary output and finally incorporating spatial diffusion.

## 2 Economic principles and basic concepts

To understand our model, we first need to establish what we mean by productivity and agglomeration. Productivity, in economic terms, refers to the rate at which goods and services are produced per unit of input, such as per worker or per hour of labour. Agglomeration economics, a field pioneered by Marshall [1], examines how geographic concentration of economic activity creates benefits that increase productivity for all participants in that region.

The principle underlying agglomeration is straightforward but powerful. Around a region of high productivity and connectivity, the productivity of each person in the surroundings increases over time [1, 6]. This increase follows a mathematical pattern: the rate of productivity increase is proportional to the current level of productivity. Regions that are already productive become even more productive at a faster rate. This concept, while intuitive from an economic standpoint, can also be formulated elegantly in mathematics.

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Consider a productivity function defined over a flat coordinate plane, which varies with time, denoted as $P = P(x, y, t)$. Here, $x$ and $y$ represent spatial coordinates indicating position on the region of interest in the plane and $t$ represents time. A simple model that captures the proportional growth behaviour we have described is given by the partial differential equation

$$\frac{\partial P}{\partial t} = kP\,, \quad k > 0 \tag{1}$$

where at time $t = 0$, we have the initial productivity distribution $P = \phi(x, y)$. The constant $k > 0$ represents the proportionality factor, essentially measuring how strongly current productivity drives future growth. This equation states that the rate of change of productivity with respect to time is directly proportional to the productivity itself.

Since the partial differential equation (1) contains no explicit dependence on the spatial variables $x$ or $y$, we can employ a clever technique from the theory of ordinary differential equations [4]. We treat it temporarily as an ordinary differential equation in $t$ alone, solve it, and then incorporate our spatial function $\phi(x, y)$ appropriately. The ordinary differential equation $\frac{dP}{dt} = kP$ has the well-known solution

$$P = Ue^{kt}, \quad U, k > 0\,.$$

By incorporating the initial condition that $P(x, y, 0) = \phi(x, y)$, we conclude that

$$P(x, y, t) = \phi(x, y)e^{kt} \tag{2}$$

where $\phi(x, y)$ represents the initial spatial distribution of productivity at time $t = 0$.

However, the basic model (2) contains a significant flaw that becomes apparent when we consider real-world constraints. It assumes that productivity can exponentially increase forever. This assumption is unrealistic for numerous reasons. First, there is finite material availability for infrastructure projects. Natural resources required for goods become scarce and expensive to procure. The supply of labour, even with population growth and migration is limited. The scale of businesses involved in a region can only expand so far before encountering diminishing returns [6]. Finally, the size of the target market for goods and services, while large, is not infinite, due to population demographic size.

These factors suggest that productivity growth must eventually slow down as it approaches some theoretical maximum. The simple exponential model fails to capture this saturation effect. This realisation leads us naturally to a more sophisticated model that incorporates growth constraints.

## 3 The logistic growth model

To address the limitations of unlimited growth, we turn to the logistic growth model, first developed by Pierre-François Verhulst in 1838 [2] to model population growth. The logistic model incorporates a carrying capacity, representing the maximum sustainable productivity level. In our context, we denote this maximum rate of productivity growth by $\eta > 0$.

Let the rate of productivity change be denoted by

$$A = \frac{\partial P}{\partial t}\,.$$

The logistic growth model modifies our earlier model (1) by introducing a factor that reduces the growth rate as we approach the maximum. We formulate this as the partial differential equation

$$\frac{\partial A}{\partial t} = kA(\eta - A) \tag{3}$$

or, in terms of $P$,

$$\frac{\partial^2 P}{\partial t^2} = k\,\frac{\partial P}{\partial t}\left(\eta - \frac{\partial P}{\partial t}\right).$$

One interpretation of (3) is as follows. When productivity is low relative to $\eta$, the value $\eta - A$ is close to $\eta$, and growth proceeds nearly as fast as in the exponential case. However, as productivity approaches $\eta$, the factor $\eta - A$ shrinks toward zero, naturally slowing the growth rate. The result is an sigmoid-shaped growth curve rather than unbounded exponential growth.

We employ the same technique as the basic model, solving the equation as if it were an ordinary differential equation [4] and later incorporating the spatial function $\phi(x, y)$ through the initial condition.

**Lemma 1.** *The solution to the ordinary differential equation*

$$\frac{dA}{dt} = kA(\eta - A) \tag{4}$$

*is* $A = \eta\big(1 + Be^{-\eta k t}\big)^{-1}$ *for some constant* $B$.

*Proof.* From (4), we perform separation of variables to obtain

$$\frac{dA}{A(\eta - A)} = k\,dt\,.$$

Using the partial fraction decomposition

$$\frac{1}{A(\eta - A)} = \frac{1}{\eta}\left(\frac{1}{A} + \frac{1}{\eta - A}\right),$$

we get that

$$\frac{1}{\eta}\int\left(\frac{1}{A}+\frac{1}{\eta-A}\right)dA=\int k\,dt$$

which yields

$$\frac{1}{\eta}\ln\left|\frac{A}{\eta-A}\right|=kt+C$$

for some constant $C$. By exponentiating both sides and solving for $A$, we obtain

$$A=\frac{\eta}{1+Be^{-\eta kt}}=\frac{\eta e^{\eta kt}}{e^{\eta kt}+B}$$

where $B=e^{\eta C}$ is another constant. $\square$

Since the expression for $A$ contains no dependence on the spatial variables $x$ or $y$, we can once again convert this to an ordinary differential equation

$$\frac{dP}{dt}=\frac{\eta e^{\eta kt}}{e^{\eta kt}+B}\,. \tag{5}$$

We obtain that

$$\int dP=\frac{1}{k}\int\frac{\eta k e^{\eta kt}}{e^{\eta kt}+B}\,dt$$

which leads to

$$P=\frac{1}{k}\ln(e^{\eta kt}+B)+C_1 \tag{6}$$

where $C_1$ is another constant of integration.

To incorporate the spatial dependence, we multiply by $\phi(x,y)$ as before to get

$$P(x,y,t)=\phi(x,y)\left(\frac{1}{k}\ln(e^{\eta kt}+B)+C_1\right).$$

We now apply the initial condition $P(x,y,0)=\phi(x,y)$. This means the bracketed expression must equal one when $t=0$. Setting $t=0$, we find

$$C_1=1-\frac{1}{k}\ln(B+1)\,.$$

Substituting this back into our expression for $P(x,y,t)$, we obtain

$$P(x,y,t)=\frac{1}{k}\phi(x,y)\left[k+\ln\left(\frac{B+e^{\eta kt}}{B+1}\right)\right]\,.$$

This is our logistic growth model for productivity, incorporating both the initial spatial distribution $\phi(x,y)$ and the saturation effect encoded by the maximum growth rate $\eta$.

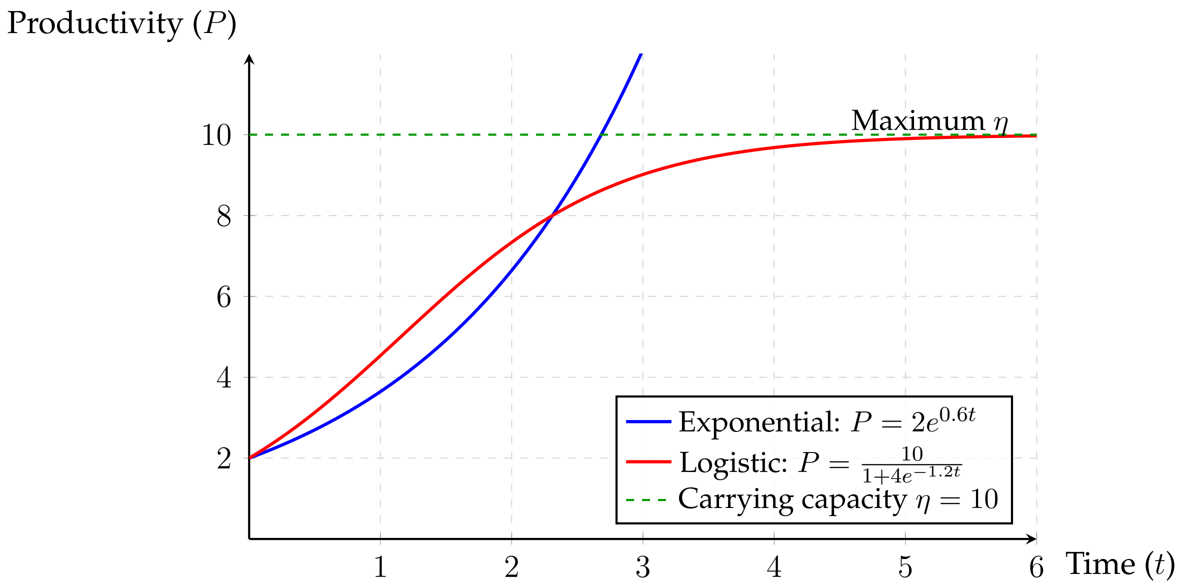


Figure 1: Comparison of exponential and logistic growth models. The exponential model (blue) exhibits unlimited growth, while the logistic model (red) approaches a maximum carrying capacity $\eta$ (green dashed line).

Figure 1 illustrates the fundamental difference between exponential and logistic growth. Initially, when productivity is far below the carrying capacity, both models predict similar growth rates. However, as time progresses, the former grows without bound, while the logistic model gradually approaches $\eta$. This S-shaped sigmoid curve of the logistic model reflects the economic reality that growth inevitably encounters constraints, reflecting the time taken for a region to establish itself economically after founding and the maximum capacity it can produce.

# 4 Calculating total monetary output

Having developed models for how productivity $P(x, y, t)$ evolves at each point in space and time, we now ask a natural economic question: what is the total monetary output of an entire region? Since productivity varies from point to point, we cannot simply multiply a single value by the area. Instead, we must integrate the productivity function over the region of interest, effectively summing up the contributions from every infinitesimal patch of land.

Mathematically, the total monetary output is given by the volume under the surface defined by $P(x, y, t)$ over a specified region. If our region extends from $x_1$ to $x_2$ in the $x$-direction and from $y_1$ to $y_2$ in the $y$-direction, then the total output at time $t = t_n$ is

$$T_n = \int_{x_1}^{x_2} \int_{y_1}^{y_2} P(x, y, t_n) \, dy \, dx \, .$$

This double integral computes the sum of productivity across the entire spatial domain at the specified moment in time. The total monetary output is generally used by economic analysts as a difference between two periods of time. This is given by

$$\Delta T = T_2 - T_1 = \int_{x_1}^{x_2} \int_{y_1}^{y_2} (P(x, y, t_2) - P(x, y, t_1))\, dy\, dx\,.$$

Suppose $P(x, y, t) = \phi(x, y)f(t)$ where $f(t)$ depends on the choice of the first or second model. By factoring out the time-dependent part, we obtain

$$\Delta T = \int_{x_1}^{x_2} \int_{y_1}^{y_2} (f(t_2) - f(t_1))\phi(x, y)\, dy\, dx\,.$$

This equation shows that the change in total output is proportional to $\phi(x, y)$ and is weighted by the temporal change factor $f(t_2) - f(t_1)$, integrated over the entire region.

# 5 Extension to include spatial diffusion

Our models thus far have captured how productivity grows over time at each fixed location, but they have neglected an important economic reality: productivity does not change in isolation at each point but rather diffuses across space. Spatial diffusion occurs due to various economic mechanisms that cause productivity gains in one location to spread to neighbouring areas [5, 7].

Consider goods and services that flow between locations. When materials are delivered or services are provided to one location, they can increase the productivity of businesses present in that region. If an electric drill is purchased and transported to a construction site, then the productivity of that site increases immediately (note that in the transferring of the drill, the seller loses the productivity gain that the drill had to offer). Similarly, the monetary value of a house appreciates when the surrounding properties increase in value, creating a spatial coupling between nearby locations. These effects suggest that we should add a term to our differential equation that accounts for how productivity spreads across the landscape.

To incorporate spatial diffusion effects, we rearrange our logistic growth model PDE and then add a spatial diffusion term reminiscent of those appearing in the heat equation [3]. Recall our differential equation under the logistic model

$$\frac{\partial A}{\partial t} = kA(\eta - A) = -k\left(\left(A - \frac{\eta}{2}\right)^2 - \frac{\eta^2}{4}\right)$$

where the last equation is found by completing the square. By isolating the quadratic term, we obtain

$$-\frac{1}{k}\frac{\partial A}{\partial t} = \left(A - \frac{\eta}{2}\right)^2 - \frac{\eta^2}{4}\,.$$

To make $A$ the subject, we isolate the squared term and take the square root. Since $A$ represents a rate of change of productivity and should be non-negative in our growth scenario, we take the positive root:

$$A - \frac{\eta}{2} = \sqrt{\frac{\eta^2}{4} - \frac{1}{k}\frac{\partial A}{\partial t}}$$

or, in terms of $P$,

$$\frac{\partial P}{\partial t} = \frac{\eta}{2} + \sqrt{\frac{\eta^2}{4} - \frac{1}{k}\frac{\partial^2 P}{\partial t^2}}\,.$$

Now we introduce the spatial diffusion term. Let $\alpha$ denote the net coefficient of spatial diffusion effects, measuring how strongly productivity spreads between neighbouring locations. The spatial diffusion contribution is given by the Laplacian of $P$ with respect to the two spatial variables, namely

$$\nabla^2 P = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\,.$$

This term captures how productivity at a point is influenced by the productivity distribution in its immediate neighbourhood [4]. If productivity is higher in surrounding areas, then this term contributes positively, pulling the local productivity upward. Conversely, if the local productivity is higher than its surroundings, then diffusion tends to spread some of that productivity outward.

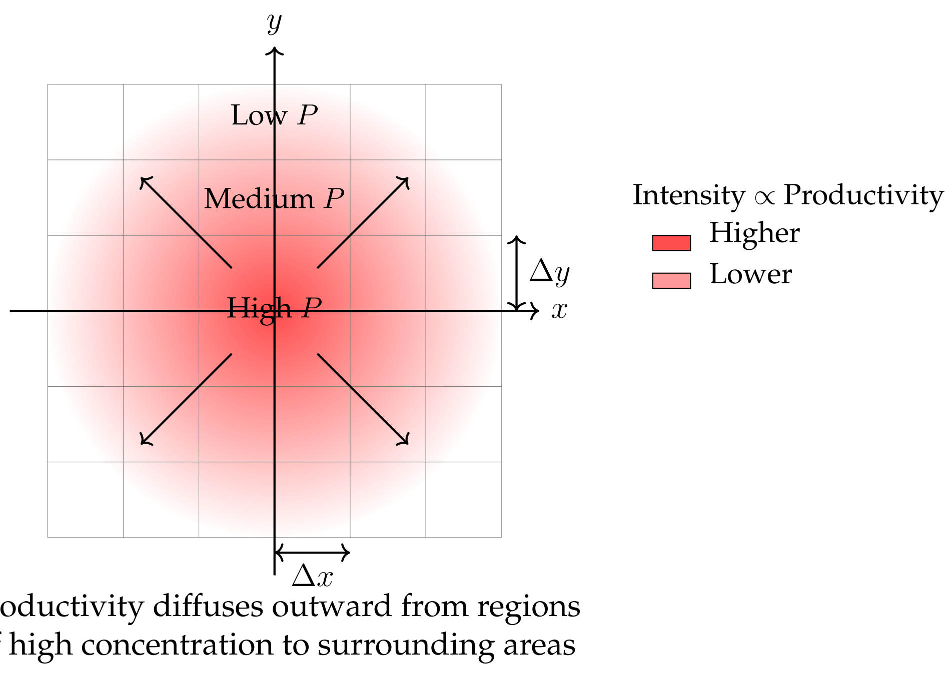


Figure 2: Visualisation of spatial diffusion of productivity.

Figure 2 provides a visual representation of the spatial diffusion process. The central region, characterised by high productivity, acts as a source from which economic benefits spread to neighbouring areas. The intensity of the colour indicates the productivity level, with red representing high productivity and transitioning through yellow to white for low productivity. The arrows demonstrate that diffusion naturally flows from regions of high productivity to regions of low productivity [6], driven by the mechanisms discussed earlier such as goods transfer, service provision, and property value appreciation.

Adding this spatial diffusion term to our temporal growth equation and rearranging, we arrive at the following nonlinear second-order partial differential equation

$$\frac{\partial P}{\partial t} = \underbrace{\frac{\eta}{2} + \sqrt{\frac{\eta^2}{4} - \frac{1}{k}\frac{\partial^2 P}{\partial t^2}}}_{\text{Controlled increase due to time}} + \underbrace{\alpha\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right)}_{\text{Spatial diffusion factors}} \tag{7}$$

with the initial condition $P(x, y, 0) = \phi(x, y)$. Here, $\eta$ represents the maximum growth rate, $k$ is the proportionality constant governing the strength of the logistic effect, and $\alpha$ is the net coefficient of spatial diffusion effects. This equation beautifully encapsulates both the temporal dynamics of productivity growth and the spatial coupling between neighbouring regions.

Unfortunately, despite considerable effort on my behalf, a general analytical solution to (7) remains elusive. The nonlinearity introduced by the square root term, combined with the coupling between temporal and spatial derivatives, places (7) beyond the reach of standard solution techniques [3, 4]. This represents a fascinating open problem for future investigation.

# 6 Conclusion

The model presented in this article illustrates how partial differential equations can capture essential features of economic productivity dynamics. Beginning with a simple exponential growth model, we recognised its limitations and developed a more realistic logistic growth framework that incorporates natural constraints on productivity increases. We then extended this model to include spatial diffusion effects by allowing productivity changes in one location to influence neighbouring areas [6].

Throughout this development, we have employed techniques from the theory of differential equations [3, 4], including separation of variables, partial fraction decomposition, and the method of integrating factors. These mathematical tools, when combined with economic intuition about agglomeration [1, 5] and diffusion, yield models that are both tractable and insightful. The progression from unlimited exponential growth through constrained logistic growth to spatially coupled diffusion represents increasing levels of complexity.

The final partial differential equation we derived, while currently lacking a general analytical solution, points toward exciting avenues for future research. More sophisticated models could incorporate temporal changes in the spatial diffusion coefficient, stochastic perturbations representing economic shocks and random fluctuations, and dynamic adjustments to the initial productivity distribution as the economic landscape evolves. Additionally, numerical methods [4] could be employed to solve the full non-linear equation for specific initial conditions, yielding quantitative predictions about productivity evolution in concrete scenarios.

# 7 Discussion

## 7.1 Model assumptions and constraints

Three fundamental constraints shape the applicability of our approach. First, the initial productivity function $\phi(x, y)$ must remain non-negative throughout the spatial domain, since negative productivity lacks economic meaning. This requirement ensures our solutions stay physically interpretable as the system evolves. In practical applications, accurately determining $\phi(x, y)$ would demand extensive empirical data collection and careful econometric analysis using surfaces of best fit to capture the true spatial distribution of economic activity.

Second, our deterministic framework treats productivity evolution as following predictable mathematical laws, but real economic systems experience constant random perturbations from countless sources including natural disasters, political shifts, technological breakthroughs, and changing consumer preferences. The models we have developed should therefore be understood as describing expected or average behaviour rather than exact trajectories. Extending this work to incorporate stochastic differential equations would allow us to model these random shocks explicitly, bringing the framework closer to capturing genuine economic complexity.

Third, our analysis assumes productivity either grows or remains constant, but economic history shows that regions frequently experience productivity declines due to resource exhaustion, recessions, or structural economic changes. A comprehensive model would need mechanisms to represent these negative scenarios, perhaps through negative growth constants or additional equation terms that activate during contractionary periods. The transitions between growth and decline regimes, such as occur during business cycles, would require time-varying parameters or switching mechanisms that alter the governing equations [5].

## 7.2 Limitations and future research directions

The non-linear partial differential equation we derived for combined temporal growth and spatial diffusion currently lacks a general analytical solution. This challenge opens doors for productive future investigation. Numerical methods, particularly finite difference and finite element approaches, could provide approximate solutions for specific parameter values and initial conditions, allowing researchers to visualise productivity evolution under various scenarios and compare model predictions with empirical observations.

Several promising extensions could make the model more realistic. The diffusion coefficient $\alpha$ might vary across space and time to reflect regional differences in transportation infrastructure, communication networks, and trade barriers. Similarly, the carrying capacity $\eta$ could become spatially dependent to capture variations in resource availability and market size across different locations [5, 7]. The implementation certain wave characteristics might better represent situations where productivity changes propagate at finite velocity rather than diffusing instantaneously. Most importantly, empirical validation through comparison with actual regional economic data would allow statistical parameter estimation and hypothesis testing, transforming these mathematical constructs from elegant abstractions into practical analytical tools. After all, if economics were entirely predictable, then we would be much richer than we are now.

# Glossary

**Agglomeration**: The geographic concentration of economic activity in a region, where the presence of firms, workers, and infrastructure creates mutual benefits that increase overall productivity.

**Carrying capacity**: In the context of logistic growth, the maximum sustainable level that a growing quantity can reach, determined by environmental or systemic constraints.

**Diffusion**: The process by which quantities spread from regions of high concentration to regions of low concentration, governed mathematically by the Laplacian operator.

**Exponential growth**: A pattern of growth where the rate of increase is proportional to the current value, leading to accelerating expansion over time, mathematically described by $\frac{dy}{dt} = ky$.

**Initial condition**: The specified value of a function at the starting point of a differential equation problem, used to determine the particular solution from the general solution.

**Laplacian**: A differential operator given by the sum of second partial derivatives, $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ in two dimensions, representing the divergence of the gradient.

**Logistic growth**: A pattern of growth that initially resembles exponential growth but slows as it approaches a maximum carrying capacity, producing an S-shaped curve, mathematically described by $\frac{dy}{dt} = ky(\eta - y)$ where $\eta$ is the carrying capacity.

**Partial differential equation**: An equation involving partial derivatives of a function with respect to multiple independent variables, used to model phenomena that vary in both space and time.

**Partial fraction decomposition**: A technique for breaking a rational function into simpler fractions that can be integrated more easily, particularly useful for integrands of the form $\frac{1}{(x-a)(x-b)}$.

**Productivity**: In economics, a measure of the efficiency of production, typically expressed as output per unit of input such as output per worker or per hour of labour.

**Proportionality constant**: A constant $k$ in an equation of the form $y = kx$ or $\frac{dy}{dt} = ky$ that determines the strength of the relationship between variables.

**Separation of variables**: A method for solving differential equations by rearranging the equation so that each variable and its differential appear on opposite sides, allowing integration of each side independently.

**Sigmoid curve**: An S-shaped curve that starts with slow growth, accelerates through a middle period, and then levels off as it approaches a maximum, characteristic of logistic growth.

**Spatial diffusion**: The spreading of a quantity across space over time, often modelled by adding Laplacian terms to differential equations to capture how neighbouring regions influence each other.

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## References

[1] A. Marshall, *Principles of Economics*, Macmillan and Company, London, 1890.

[2] P.-F. Verhulst, Notice sur la loi que la population suit dans son accroissement, *Correspondance Mathématique et Physique* **10** (1838), 113–121.

[3] L.C. Evans, *Partial Differential Equations*, American Mathematical Society, 2010.

[4] W.A. Strauss, *Partial Differential Equations: An Introduction*, John Wiley & Sons, 2008.

[5] P. Krugman, Increasing returns and economic geography, *Journal of Political Economy* **99** (1991), 483–499.

[6] M. Whearty, Seminar on the effects of inequalities in housing development and transport provisions on socioeconomic disparities in England and Wales, University of Sydney (2025).

[7] M. Fujita, P. Krugman and A.J. Venables, *The Spatial Economy: Cities, Regions, and International Trade*, MIT Press, 1999.