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# Modeling Kepler's planetary motion on an elliptical orbit using an orthographic projection

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## 1 Introduction

By means of observational data, Kepler determined that the shape of planetary orbits in the solar system is rather simple: an ellipse with the sun at one of the foci. From the data, he made a mathematical model of the planetary motion and came up with three laws of its movement.

In this article, I attempt to derive a mathematical model of planetary motion solely from Kepler's three laws of movement without using any observational data. This approach would be rather theoretical but there are a few lessons to learn. By choosing a suitable orthographic projection, the elliptical trajectory nature of a planetary motion can be projected to a circular orbit. The idea is that this projected circular orbit simplifies the analysis and the required computation.

## 2 Modeling Kepler's planetary motion

An orthographic projection is essentially a way to represent three-dimensional object as a two-dimensional object by using parallel lines that are perpendicular to the projection plane.

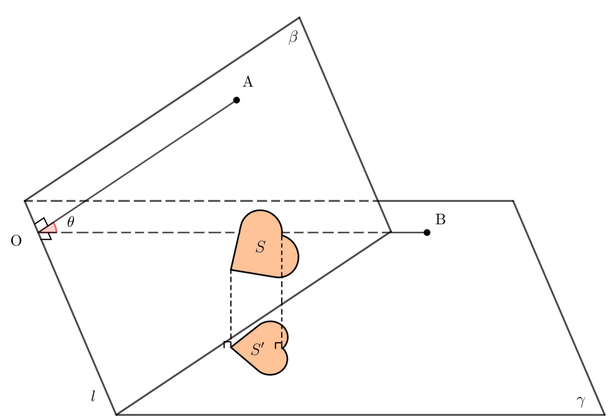


Figure 1: Orthographic projection of the object on the plane $\beta$ to the plane $\gamma$

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Let $\beta$ be a plane in a 3-dimensional space and $S \subseteq \beta$ be a region. Draw a perpendicular from each point on the subject to another plane $\gamma$ with intersection line $l$. This set of the foot of perpendiculars forms an image of the orthographic projection. If the dihedral angle between the two planes $\beta$ and $\gamma$ is $\theta$, then the projected image $S'$ shrinks in area with

$$\operatorname{area}(S') = \operatorname{area}(S) \cos\theta \, .$$

This is illustrated in Figure 1.

## 2.1 Kepler's first two laws of planetary motion

Johannes Kepler (1571–1630) was a contemporary of Galileo Galilei (1564–1642) who excelled at using mathematical and physical approaches to scrutinise the data obtained by observing the sky. After his mentor Tycho Brahe (1546–1601) died, Kepler inherited volumes of observational data and continued his research on the movement of Mars. He proposed laws of planetary motion, and the first two of them [2, pp. 64–93] read as follows:

(1) The planets move in elliptical orbits, with the sun at one focus.

(2) The line connecting the planet and the sun sweeps out equal areas in equal times.

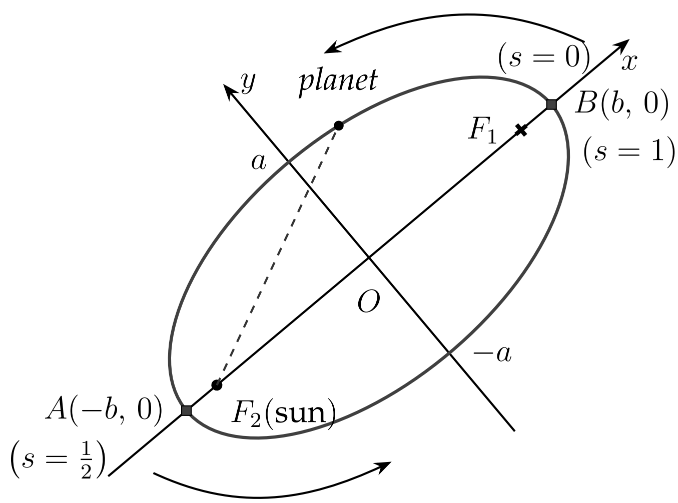


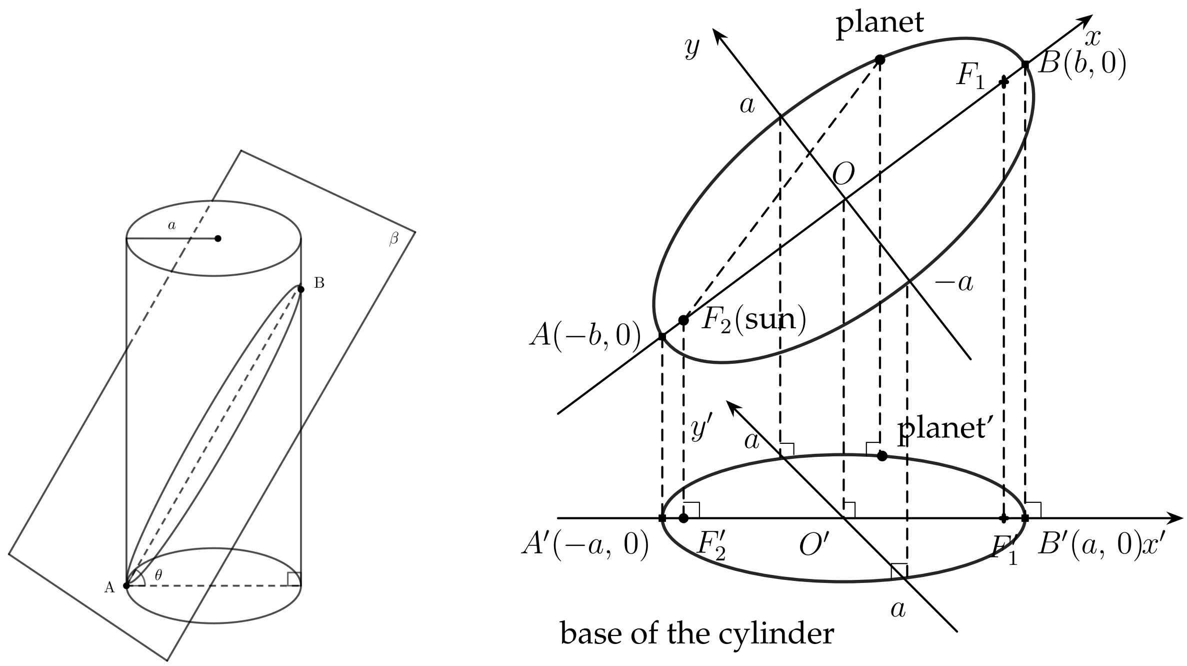

Figure 2: The elliptical orbit of a planet, where $F_1$ and $F_2$ denote its foci

Let $a$ and $b$ be the length of the semi-minor axis and semi-major axis in a planet's elliptical orbit, respectively, with $0 < a < b$. It is known that the two foci are located on the major axis with distance $\sqrt{b^2 - a^2}$ from the centre, but in the opposite direction. The *eccentricity* $\tilde{e}$ of the elliptical orbit ellipse is

$$\tilde{e} := \sqrt{1 - \frac{a^2}{b^2}} \, .$$

As $\tilde{e} \to 0$, the ellipse will look more similar to a circle and indeed when $\tilde{e} = 0$, we get a circle. Here, the notation $\tilde{e}$ is used to denote eccentricity to separate it from the Euler's number $e$ to reduce confusion.

Let us consider the intersection region between a right cylinder and a plane. The cylindrical section [7] is the intersection of a cylinder's surface with a plane. When a plane meets the base of a right cylinder at one point and the angle between the plane and the base of the cylinder is acute, the cylindrical section is an ellipse.




(a) A cylindrical section (b) Orthographic projection of the ellipse on $\beta$ to the base

Figure 3: The image of an orthographic projection of the elliptical orbit is a circle

Put the $xy$-axes on the plane $\beta$ containing the elliptical orbit of this planet so that the major axis is on the $x$-axis and the minor axis is on the $y$-axis with the centre of its orbit at the origin. Then the orbit's image of the orthographic projection is the circle on the base with radius $a$, which is the length of the orbit's semi-minor axis. If $\theta$ is the angle between the plane and the base of the cylinder, then $\cos\theta = a/b$. Since the foci are away from the centre with an equal distance $\sqrt{b^2 - a^2}$, the $xy$-coordinates for $F_2$ and its orthographic image $F_2'$ on the base in $x'y'$-coordinates are

$$F_2 = (-\sqrt{b^2 - a^2}, 0), \quad F_2' = (-\sqrt{b^2 - a^2}\cos\theta, 0) = (-a\tilde{e}, 0)\,.$$

## 2.2 Assumptions

Let $\alpha : [0, \infty) \to [0, \infty)$ be a function that measures the angle between $O'P'$ and $O'B'$, where $O'$ is the origin, $P'$ is the orthographic image of the planet $P$ and $B'$ is the orthographic image of the aphelion $B$. This is illustrated in Figure 4. For simplicity, we write $\alpha(s) = \alpha_s$. Here, $s$ is dimensionless.

Furthermore, we assume that $\alpha$ satisfies the following properties.

(a) The function $\alpha$ is continuously differentiable on $(0, \infty)$ to model the smooth motion of the planet without a sudden change of its movement due to an external force.

(b) The function $\alpha_s$ is strictly increasing; that is, $\alpha_{s_1} < \alpha_{s_2}$ whenever $s_1 < s_2$. In other words, the planet is assumed to never stop moving.

(c) The function $\alpha$ satisfies the following recurrence:

$$\alpha_{s+1} = 2\pi + \alpha_s \ \text{ for } s \geq 0\,, \qquad \alpha_0 = 0\,, \ \ \alpha_{\frac{1}{2}} = \pi\,, \ \ \alpha_1 = 2\pi\,.$$

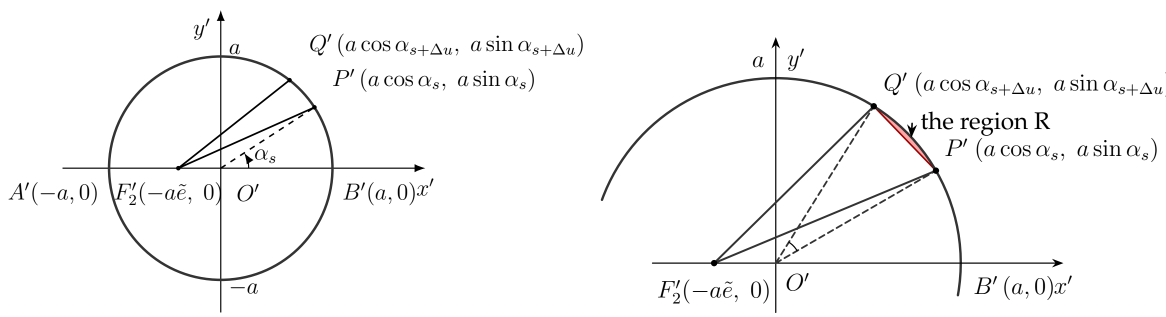



Figure 4: Two points on the orthographic image when their $y'$-coordinates are positive

Take another dimensionless variable $u \geq 0$ that does not depend on $s$ such that the area of the sector $F_2'P'Q'$ only depends on $\Delta u > 0$. In other words,

$$\frac{d}{ds}(\text{area of the sector } F_2'P'Q') = 0\,.$$

The sector $F_2'P'Q'$ can be separated into two parts: the triangle $\Delta F_2'P'Q'$ and the region $R$. For a natural number $n$, we assume that $\Delta u$ is small enough such that

$$2(n-1)\pi \leq \alpha_s < \alpha_{s+\Delta u} \leq (2n-1)\pi\,.$$

Recall that the area $T$ of the triangle whose vertices are $(0,0)$, $(a_1, a_2)$ and $(b_1, b_2)$ is

$$T = \frac{|a_1 b_2 - a_2 b_1|}{2}\,. \tag{1}$$

Using the formula (1) with the coordinates of $F_2', Q', P'$ in Figure 4, we have that

$$\overrightarrow{F_2'P'} = \begin{pmatrix} a(\tilde{e} + \cos\alpha_s \\ a\sin\alpha_s \end{pmatrix}, \quad \overrightarrow{F_2'Q'} = \begin{pmatrix} a(\tilde{e} + \cos\alpha_{s+\Delta u}) \\ a\sin\alpha_{s+\Delta u} \end{pmatrix}$$

and

$$\text{area}(\Delta F_2'P'Q') = \frac{1}{2}\left|a^2(\tilde{e} + \cos\alpha_s)\sin\alpha_{s+\Delta u} - a^2(\tilde{e} + \cos\alpha_{s+\Delta u})\sin\alpha_s\right|$$

$$= \frac{a^2}{2}\left|\, \sin\left(\alpha_{s+\Delta u} - \alpha_s\right) + \tilde{e}\left(\sin \alpha_{s+\Delta u} - \sin \alpha_s\right) \right|$$
$$= \frac{a^2}{2}\left( \sin\left(\alpha_{s+\Delta u} - \alpha_s\right) + \tilde{e}\left(\sin \alpha_{s+\Delta u} - \sin \alpha_s\right) \right).$$

To justify the last equation, note that because $\cos$ is decreasing on $[2(n-1)\pi, (2n-1)\pi]$, we must have, on two intervals of the same length $\alpha_{s+\Delta u} - \alpha_s$, that

$$\sin\left(\alpha_{s+\Delta u} - \alpha_s\right) - \left(\sin \alpha_{s+\Delta u} - \sin \alpha_s\right) = \int_0^{\alpha_{s+\Delta u}-\alpha_s} \cos x \, dx - \int_{\alpha_s}^{\alpha_{s+\Delta u}} \cos x \, dx > 0\,.$$

The area of $R$ can be obtained from the area of the sector $O'P'Q'$ by subtracting the area of the triangle O′P′Q′ as follows:

$$\begin{aligned} \text{area}(R) &= \frac{1}{2}a^2(\alpha_{s+\Delta u} - \alpha_s) - \frac{1}{2}a^2 \sin(\alpha_{s+\Delta u} - \alpha_s) \\ &= \frac{1}{2}a^2\big(\alpha_{s+\Delta u} - \alpha_s - \sin(\alpha_{s+\Delta u} - \alpha_s)\big)\,. \end{aligned}$$

Therefore, the area of the sector $F_2'P'Q'$ is

$$\text{area}(\text{sector } F_2'P'Q') = \frac{a^2}{2}\left(\alpha_{s+\Delta u} - \alpha_s + \tilde{e}\left(\sin \alpha_{s+\Delta u} - \sin \alpha_s\right)\right) := W(s, \Delta u)\,. \quad (2)$$

According to Kepler's Law of Motion, the quantity $W(s, \Delta u)$ should be independent of the variable $s$, even though $s$ is present in the formula. By some algebraic manipulation,

$$\begin{aligned} W(s, \Delta u) &= \frac{a^2}{2}\left(\alpha_{s+\Delta u} - \alpha_s + \tilde{e}\left(\sin \alpha_{s+\Delta u} - \sin \alpha_s\right)\right) \\ &= \frac{a^2 \Delta u}{2}\left(\frac{\alpha_{s+\Delta u} - \alpha_s}{\Delta u} + \frac{\sin \alpha_{s+\Delta u} - \sin \alpha_s}{\alpha_{s+\Delta u} - \alpha_s}\frac{\alpha_{s+\Delta u} - \alpha_s}{\Delta u}\tilde{e}\right). \end{aligned}$$

By our assumption, the factor inside the round bracket in the last equation above is independent of $s$. By taking the limit as $\Delta u \to 0^+$, we see that

$$W(s, \Delta u) \approx \frac{a^2 \Delta u}{2}\left(\alpha_s' + \tilde{e}\alpha_s' \cos \alpha_s\right)\,.$$

Since $\alpha_s' + \tilde{e}\alpha_s' \cos \alpha_s$ does not depend on $\Delta u$, it is constant. By letting $k$ to be this constant, we have that

$$\frac{d}{ds}\left(\alpha_s + \tilde{e}\sin \alpha_s\right) = \alpha_s' + \tilde{e}\alpha_s' \cos \alpha_s = k\,,$$

and by integrating both sides with respect to $s$, we have that

$$\alpha_s + \tilde{e}\sin \alpha_s = ks + C$$

for some constant $C$. To determine the values of $k$ and $C$, we use the conditions $\alpha_0 = 0$ and $\alpha_{\frac{1}{2}} = \pi$. When $s = 0$, the equation becomes

$$\alpha_0 + \tilde{e}\sin\alpha_0 = C$$

and this leads to $C = 0$. Similarly, when $s = \frac{1}{2}$, we obtain $k = 2\pi$.

Therefore, for $s \geq 0$, we have that

$$\alpha_s + \tilde{e}\sin\alpha_s = 2\pi s\,. \tag{3}$$

By applying (3) to (2), we obtain

$$\begin{aligned} W(s, \Delta u) &= \frac{a^2}{2}\big((\alpha_{s+\Delta u} + \tilde{e}\sin\alpha_{s+\Delta u}) - (\alpha_s + \tilde{e}\sin\alpha_s)\big) \\ &= \frac{a^2}{2}\big(2\pi(s+\Delta u) - 2\pi s\big) \\ &= \pi a^2 \Delta u\,. \end{aligned}$$

## 2.3 The uniqueness of the solution for $W(s, \Delta u)$

Define a function $g$ on $[0, \infty)$ by $g(s) = \alpha_s + \tilde{e}\sin\alpha_s$ and a function $f$ on $\mathbb{R}$ by $f(x) = g(s+x) - g(s)$. Since $\alpha_{s+1} = 2\pi + \alpha_s$ for $s \geq 0$, we have

$$f(0) = 0\,, \quad f(1) = 2\pi\,, \qquad g(1+s+x) = 2\pi + g(s+x) \;\text{ for } s \geq 0\,.$$

This implies that

$$f(1+x) = g(s+1+x) - g(s) = 2\pi + g(s+x) - g(s) = 2\pi + f(x)\,.$$

We now consider the following one-sided limits:

$$\begin{aligned} \lim_{h\to 0^+} \frac{f(h) - f(0)}{h} &= \lim_{h\to 0+} \frac{f(1+h) - 2\pi}{h} \\ &= \lim_{h\to 0+} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0+} \frac{g(s+1+h) - g(s) - 2\pi}{h} \\ &= \lim_{h\to 0+} \frac{g(s+1+h) - g(s+1)}{h} \end{aligned}$$

and

$$\begin{aligned} \lim_{h\to 0^-} \frac{f(h) - f(0)}{h} &= \lim_{h\to 0-} \frac{f(h) + 2\pi - 2\pi}{h} \\ &= \lim_{h\to 0-} \frac{f(1+h) - f(1)}{h} \end{aligned}$$

$$= \lim_{h\to 0-} \frac{g(s+1+h)-g(s)-2\pi}{h}$$
$$= \lim_{h\to 0-} \frac{g(s+1+h)-g(s+1)}{h}\,.$$

Since $g$ is continuously differentiable on $(0,\infty)$,

$$g'(s) = \frac{d}{ds} g(s+1) = \lim_{h\to 0} \frac{g(s+1+h)-g(s+1)}{h}$$

and

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{f(h)-f(0)}{h},$$

we have that $f'(1) = g'(s) = 2\pi$ and $g(s) = 2\pi s$.

We can summarise everything in the following proposition.

**Proposition 1.** *Suppose that a planet having an orbital period $T$ moves in an elliptical orbit, whose centre is $O$ and eccentricity is $\tilde{e}$. Let $P_t$ denote the position at time $t \geq 0$, and assume that for a suitable orthographic projection, the projection image of the elliptical orbit is a circle with centre $O'$ and the projection image of $P_t$ is $P'_t$.*

*Let $\alpha_{t/T}$ denote the radian angle between the two rays $O'P'_T$ and $O'P'_t$. Then*

$$\alpha_{t/T} + \tilde{e} \sin \alpha_{t/T} = \frac{2\pi t}{T}\,.$$

## 2.4 Extending $\alpha_s$ to the whole domain

Now, we are ready to extend $\alpha$. Since $\alpha$ is a strictly increasing function, it has an inverse function $\alpha^{-1}$, with

$$\alpha^{-1}(x) = \frac{x}{2\pi} + \frac{\tilde{e}}{2\pi} \sin x \quad \text{for} \quad 2(n-1)\pi \leq x \leq (2n-1)\pi\,,\ \ n \in \mathbb{N}\,. \tag{4}$$

This is due to the relation $\alpha_s + \tilde{e} \sin \alpha_s = 2\pi s$. For $x \in ((2n-1)\pi, 2n\pi)$,

$$s = n - \alpha^{-1}(2n\pi - x)$$

is defined by the symmetry of the planetary motion.

You might have expected that $\alpha_{\frac{1}{4}} = \frac{\pi}{2}$ or $\alpha_{\frac{3}{4}} = \frac{3\pi}{2}$, but this is not true. The actual values of $s = \alpha^{-1}(x)$, when $x = \pi/2$ and $x = 3\pi/2$, are

$$\alpha^{-1}\left(\frac{\pi}{2}\right) = \frac{1}{4} + \frac{\tilde{e}}{2\pi}\,, \quad \alpha^{-1}\left(\frac{3\pi}{2}\right) = \frac{3}{4} - \frac{\tilde{e}}{2\pi}\,.$$

The gap $\pm\tilde{e}/2\pi$ depends on the eccentricity or, in other words, the shape of its elliptical orbit.

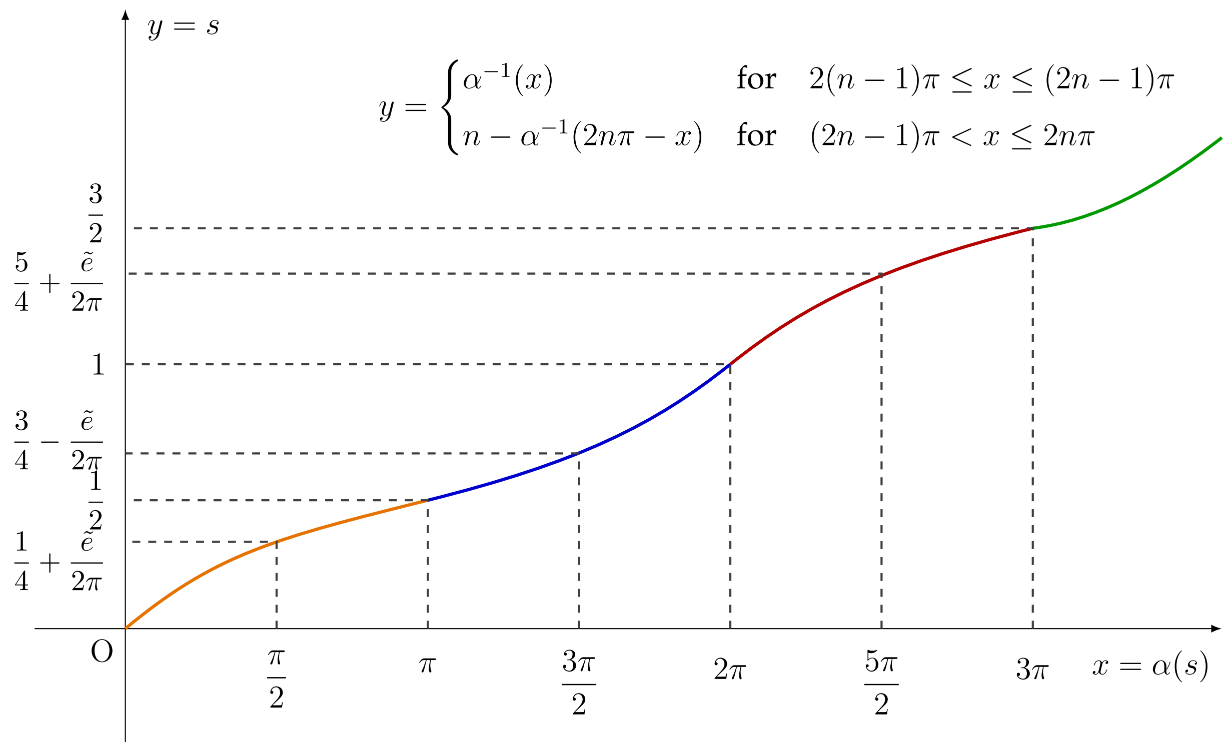



Figure 5: The plot of $\alpha^{-1}$ on $[2(n-1)\pi, 2n\pi]$

When $2(n-1)\pi \le \alpha_s < (2n-1)\pi < \alpha_{s+\Delta u} \le 2n\pi$, we can still describe the area of the sector $F_2'P'Q'$ by using the symmetry of the planet's movement. Let $A'$ be the point $(-a, 0)$ on the circle. Then the components of the vectors are

$$\overrightarrow{F_2'P'} = \begin{pmatrix} a(\tilde{e} + \cos\alpha_s) \\ a\sin\alpha_s \end{pmatrix}, \quad \overrightarrow{F_2'A'} = \begin{pmatrix} a(-1+\tilde{e}) \\ 0 \end{pmatrix}, \quad \overrightarrow{F_2'Q'} = \begin{pmatrix} a(\tilde{e} + \cos\alpha_{s+\Delta u}) \\ a\sin\alpha_{s+\Delta u}. \end{pmatrix}$$

The area of $\Delta F_2'P'A'$ is

$$\operatorname{area}(\Delta F_2'P'A') = \frac{1}{2}\left|a^2(-1+\tilde{e})\sin\alpha_s\right| = \frac{1}{2}a^2(1-\tilde{e})\sin\alpha_s\,. \tag{5}$$

Likewise,

$$\operatorname{area}(\Delta F_2'Q'A') = \frac{1}{2}\left|a^2(-1+\tilde{e})\sin\alpha_{s+\Delta u}\right| = \frac{1}{2}a^2(-1+\tilde{e})\sin\alpha_{s+\Delta u}\,. \tag{6}$$

The two areas of $R_1$ and $R_2$ can be found by using the areas of the two sectors $O'P'A'$ and $O'A'Q'$, and the two triangles $\Delta O'P'A'$ and $\Delta O'A'Q'$. More precisely,

$$\begin{aligned}\operatorname{area}(R_1) &= \operatorname{area}(O'P'A') - \operatorname{area}(\Delta O'P'A') \\ &= \frac{a^2}{2}\big((2n-1)\pi - \alpha_s - \sin\alpha_s\big)\,. \end{aligned} \tag{7}$$

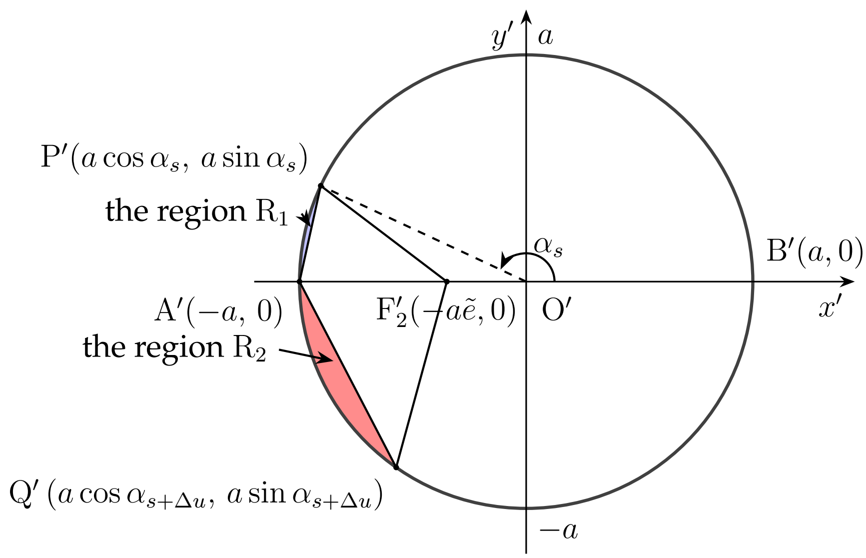



Figure 6: Illustration of the case $2(n-1)\pi \leq \alpha_s < (2n-1)\pi < \alpha_{s+\Delta u} \leq 2n\pi$

Note that, for $n \in \mathbb{N}$, we have that $\sin\alpha_{s+\Delta u} \leq 0$ for $(2n-1)\pi < \alpha_{s+\Delta u} \leq 2n\pi$ and $\alpha_{n-\frac{1}{2}} = (2n-1)\pi$. By symmetry, we have that

$$\begin{aligned}\alpha_{s+\Delta u} - (2n-1)\pi &= (2n-1)\pi - \alpha_{2n-1-s-\Delta u},\\ \sin\alpha_{s+\Delta u} &= -\sin\alpha_{2n-1-s-\Delta u}\,.\end{aligned}$$

Employing this symmetry, we find that

$$\begin{aligned}\mathrm{area}(R_2) &= \mathrm{area}(O'A'Q') - \mathrm{area}(\Delta O'A'Q')\\ &= \frac{a^2}{2}\big(\alpha_{s+\Delta u} - (2n-1)\pi\big) + \frac{a^2}{2}\sin\alpha_{s+\Delta u}\\ &= \frac{a^2}{2}\big(\alpha_{s+\Delta u} - (2n-1)\pi + \sin\alpha_{s+\Delta u}\big)\\ &= \frac{a^2}{2}\big((2n-1)\pi - \alpha_{2n-1-s-\Delta u} - \sin\alpha_{2n-1-s-\Delta u}\big). \end{aligned} \tag{8}$$

By summing up all the relevant areas from equations (5), (6), (7) and (8), we finally have that

$$\begin{aligned}W(s,\Delta u) &= \mathrm{area}(\Delta F_2'P'A') + \mathrm{area}(F_2'A'Q') + \mathrm{area}(R_1) + \mathrm{area}(R_2)\\ &= \frac{a^2}{2}\big((4n-2)\pi - (\alpha_s + \tilde{e}\sin\alpha_s) - (\alpha_{2n-1-s-\Delta u} + \tilde{e}\sin\alpha_{2n-1-s-\Delta u})\big)\\ &= \frac{a^2}{2}\big((4n-2)\pi - 2\pi s - 2\pi(2n-1-s-\Delta u)\big)\\ &= a^2\pi\Delta u\,.\end{aligned}$$

## 2.5 The velocity of the planet on its orbit

Kepler said that the planet moves the fastest around perihelion $A$ and slowest around aphelion $B$ by interpreting these laws of planetary motion [2]. The motion of the planet on the plane $\beta$ can be written in $xy$-coordinates as follows. The components of the position vector $\overrightarrow{\mathrm{OP}}$ at time $t \geq 0$ is

$$\overrightarrow{OP} = \begin{pmatrix} b\cos\alpha_s \\ a\sin\alpha_s \end{pmatrix}$$

where $s = t/T$. The velocity vector

$$\overrightarrow{v} = \frac{d}{dt}\overrightarrow{\mathrm{OP}} = \left(\frac{d}{ds}\overrightarrow{\mathrm{OP}}\right)\frac{ds}{dt} = \frac{1}{T}\begin{pmatrix} -b\sin\alpha_s \cdot \alpha_s' \\ a\cos\alpha_s \cdot \alpha_s' \end{pmatrix}$$

is obtained by taking the derivative of its position vector with respect to $t$. With this in mind,

$$\left|\overrightarrow{v}\right|^2 = \left(\frac{\alpha_s'}{T}\right)^2 \left(b^2\sin^2\alpha_s + a^2\cos^2\alpha_s\right) = \left(\frac{\alpha_s'}{T}\right)^2 \left(a^2 + (b^2 - a^2)\sin^2\alpha_s\right).$$

Since $\left|\overrightarrow{v}\right|^2$ is continuous on the closed interval $t \in [0, T]$, its maximum and minimum values are attainable. Note that from the relation $\alpha_s + \tilde{e}\sin\alpha_s = 2\pi s$ where $s = \frac{t}{T}$, we have

$$\frac{d}{dt}\alpha_s = \left(\frac{d}{ds}\alpha_s\right)\frac{ds}{dt} = \frac{1}{T}\alpha_s' = \frac{2\pi/T}{1 + \tilde{e}\cos\alpha_s}. \tag{9}$$

We use (9) to obtain

$$\frac{d}{dt}\left|\overrightarrow{v}\right|^2 = \frac{d}{d\alpha_s}\left|\overrightarrow{v}\right|^2 \frac{d\alpha_s}{dt} = \frac{8\pi^2 b^2 \tilde{e}(1 + \tilde{e}\cos\alpha_s)\sin\alpha_s}{T(1 + \tilde{e}\cos\alpha_s)^3}.$$

Since $\tilde{e} = \sqrt{1 - \frac{a^2}{b^2}} < 1$ and $1 + \tilde{e}\cos\alpha_s > 0$ for $s \geq 0$, the sign of $\frac{d}{dt}\left|\overrightarrow{v}\right|^2$ is solely determined by the sign of the $\sin\alpha_s$. Within $t \in [0, T]$, $\left|\overrightarrow{v}\right|^2$ attains its maximum value when $t = T/2$ and $\left|\overrightarrow{v}\right|^2$ attains its minimum value when $t = 0$ or $t = T$.

We summarise everything in the following proposition.

**Proposition 2.** *The quantity*

$$\frac{1 + \tilde{e}}{1 - \tilde{e}}$$

*indicates the ratio of the magnitude of the maximum velocity over the magnitude of the minimum velocity of a planet in a solar system.*

*Proof.* The proof follows by noting that

$$\left|\overrightarrow{v}_{max}\right|^2 = \left(\frac{\alpha_{1/2}'}{T}\right)^2 \left(a^2 + (b^2 - a^2)\sin^2\alpha_{1/2}\right) = \left(\frac{2\pi}{1 - \tilde{e}}\right)^2 \frac{a^2}{T^2},$$

$$\left|\overrightarrow{v}_{min}\right|^2 = \left(\frac{\alpha_1'}{T}\right)^2 \left(a^2 + (b^2 - a^2)\sin^2\alpha_1\right) = \left(\frac{2\pi}{1 + \tilde{e}}\right)^2 \frac{a^2}{T^2}$$

where $a$ is the length of the semi-minor axis of the elliptical orbit. □

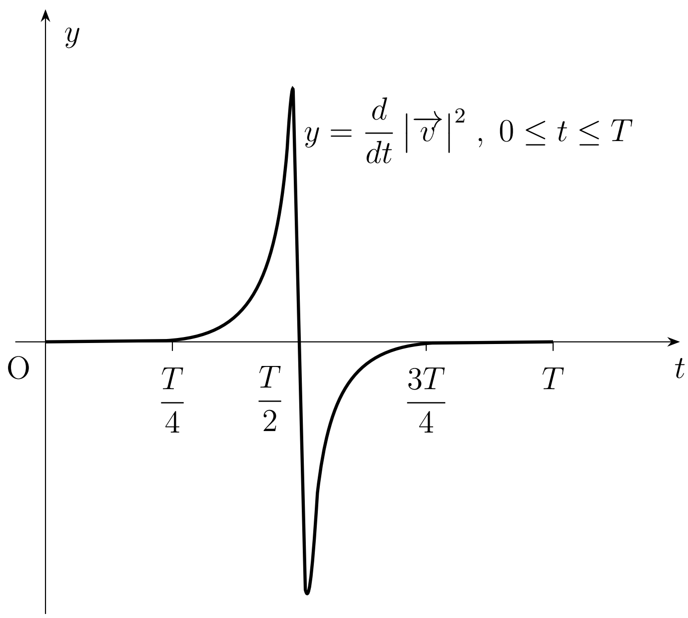



Figure 7: The sign of $\frac{d}{dt}\left|\overrightarrow{v}\right|^2$ depends on the sign of $\sin \alpha_s$

## 2.6 Applying the Third Law of Motion to this model

In $1618$, Kepler proposed the relationship between the planet's distance from the sun and its orbital period, which is called *The Third Law of Planetary Motion* in his book *Harmonices mundi libri V* (Five books on the harmony of the world) [2]. It reads:

(3) The ratio of the period squared over the distance from the sun cubed was the same for all planets.

Note that "the distance from the sun" in the third law denotes the mean distance [9], which is the length of the semi-major axis, or the average of the farthest point (aphelion) and the closest point (perihelion) of a celestial body's elliptical orbit from the sun.

In this model, the length of the semi-major axis is $b$ and the orbital period is denoted by $T$. According to the Third Law of Motion, the planetary movement in the solar system obeys

$$\frac{T^2}{b^3} = k_0 \,, \tag{10}$$

where the constant $k_0$ is approximately

$$k_0 \approx \frac{4\pi^2}{GM} \approx 1.334 \times 10^5 \, \frac{\text{days}^2}{\text{AU}^3} \approx 1 \, \frac{\text{year}^2}{\text{AU}^3}$$

(see [8] for more information). Here, AU stands for "astronomical unit", which is the length of the semi-major axis of the Earth's elliptical orbit, $G$ denotes the gravitational constant, and the $M$ denotes the mass of the sun.

By solving (10) for $T$, we get $T = \sqrt{k_0 b^3}$. Now, by letting $s = t/\sqrt{k_0 b^3}$, we get the important relation

$$\alpha_{t/\sqrt{k_0 b^3}} + \tilde{e} \sin \alpha_{t/\sqrt{k_0 b^3}} = \frac{2\pi t}{\sqrt{k_0 b^3}}$$

that summarises the three laws of Kepler's planetary motion.

# 3 Conclusion

Kepler's Second Law of Planetary Motion pertains to the area inside the elliptical orbit swept out by an equal time interval. Since the orthographic projection image preserves the inequality of areas being compared on the same plane, the relationship among areas in the second law of planetary motion still holds among its orthographically projected images. By choosing a suitable orthographic projection, the image of an elliptical orbit can be circular, and it also provides the relationship between the areas and the angular velocity of the planet in circular motion.

The Third Law of Motion gives an expression of the orbital period $T$ in terms of the length of the semi-major axis of its elliptical orbit. We then pre-project this circular orbit to its original plane, where the elliptical orbit is contained, and the modeling of the planetary motion in the solar system could be constructed. In the $xy$-coordinates, the position $P$ of the planet on its elliptical orbit at time $t \geq 0$ is

$$P = (b \cos \alpha_s, a \sin \alpha_s), \quad \text{where} \quad s = \frac{t}{\sqrt{k_0 b^3}} \quad \text{and} \quad \alpha_s + \tilde{e} \sin \alpha_s = 2\pi s\,.$$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# Appendix: A *Parabola* problem and Cavalieri's Principle

In *Parabola* Problem 1763 [1], a blue region is given as below in Figure 8 and the task is to find its area.

The setup is as follows. Consider the rectangle $PQRS$ containing two quarters of ellipses with centers at $Q$ and $R$, respectively. Let $a$ be the width and $b$ be the length of the rectangle, respectively with $0 < a < b$. This problem can be solved by some calculus, but given the symmetry, I was thinking of approaching this problem via a suitable orthographic projection, which allows for the projected image to contain a square and two arcs of circles as opposed to a rectangle and two arcs of ellipses.

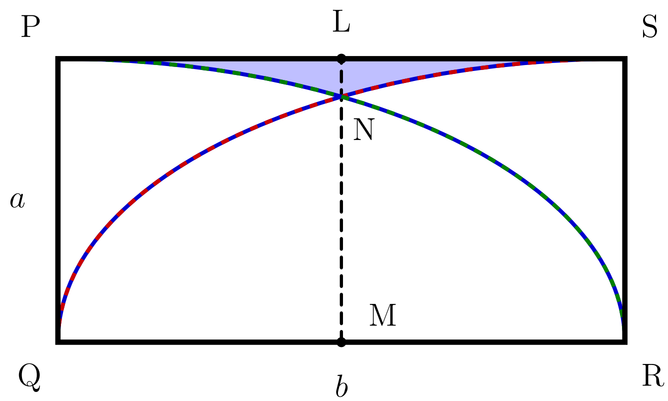
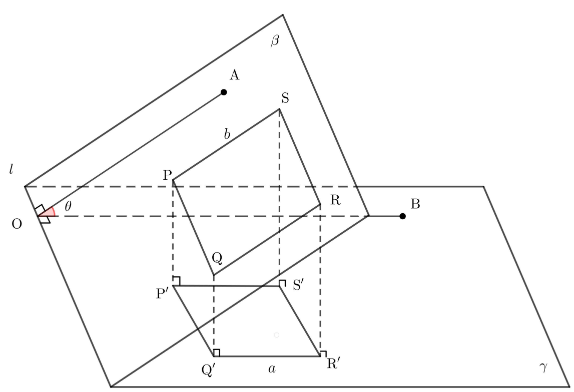
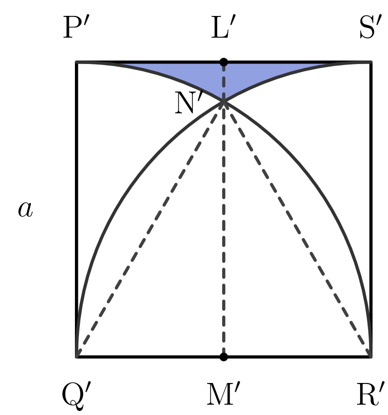



Figure 8: The blue region in the rectangle $PQRS$

Use the orthographic projection of the rectangle $PQRS$ contained in the plane $\beta$ on some other plane $\gamma$, such that the radian angle between $\beta$ and $\gamma$ is $\theta$. This means that $\cos\theta = a/b$. Both lines $PQ$ and $SR$ are parallel to the intersection line $l$ of $\beta$ and $\gamma$:




(a) The orthographic projection of the rectangle PQRS in the place $\beta$ onto the plane $\gamma$




(b) The orthographic image of the blue region is inside the square $P'Q'R'S'$.

Figure 9: A square image of a suitable orthographic projection of a rectangle

In the projected image, rectangle $P'Q'R'S'$ is square and the triangle $N'Q'R'$ is equilateral, so $\angle P'Q'N' = \pi/6$. Let $S'$ be the area of the orthographic projection image of the blue region. We subtract the area of the sectors $P'Q'N'$, $N'R'S'$ and the area of $\Delta N'Q'R'$ from the area of the square $P'Q'R'S'$ as follows:

$$S' = a^2 - \frac{1}{2}a^2 \times \frac{\pi}{6} - \frac{1}{2}a^2 \times \frac{\pi}{6} - \frac{1}{2}a^2 \times \frac{\sqrt{3}}{2} = a^2\left(1 - \frac{\pi}{6} - \frac{\sqrt{3}}{4}\right).$$

Since $S' = S \times \cos\theta$, we finally have that the original area $S$ of the blue region is

$$S = \frac{b}{a} \times a^2\left(1 - \frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) = ab\left(1 - \frac{\pi}{6} - \frac{\sqrt{3}}{4}\right).$$

## 2D Cavalieri's Principle problem

There is an extension of the Parabola problem called *Cavalieri's Principle* [5] in the 2-dimensional case. It is the case when two regions in a plane are included between two parallel lines or curves in that plane:

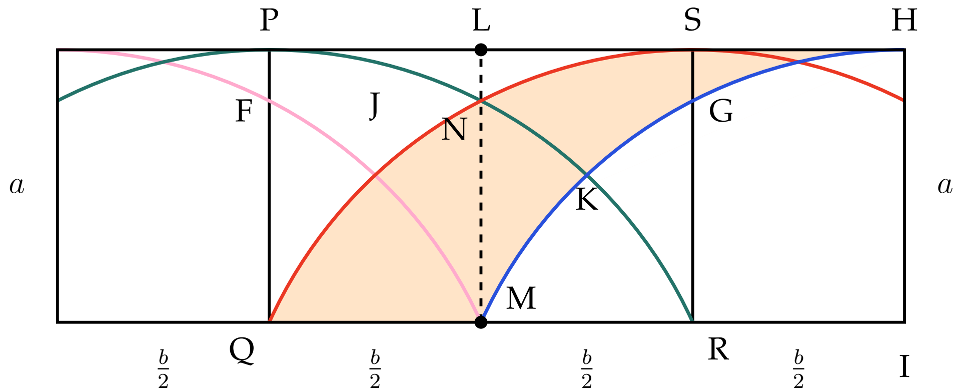
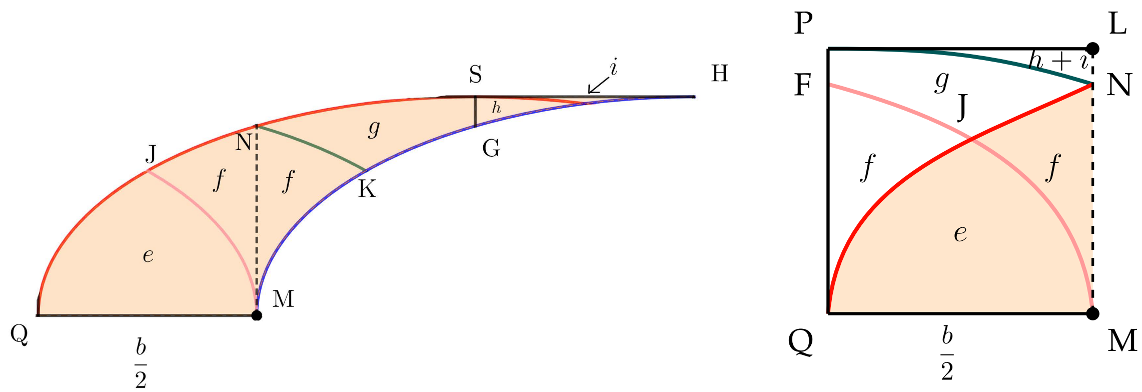



Figure 10: The area $\mathrm{SQMH}$ between the two arcs of ellipses, which are parallel translation by $\frac{b}{2}$.

Extend the original rectangle $PQRS$ to the left and right by the length $\frac{1}{2}b$ followed by parallel translation of the two quarters of the ellipses to the same direction by the same length $\frac{1}{2}b$. The orange region $SQMH$ is surrounded by the two quarters of the ellipses and the line segments $SH$ and $QM$. When the coloured region $SQMH$ is sliced by a horizontal line, the length of its line segment inside this region is $\frac{1}{2}b$. Cavalieri's Principle asserts that the area of this region is the same as that of the rectangle $\mathrm{PQML}$.




(a) The original region between the two arcs of ellipses (b) The new region after reflections and translations

Figure 11: An example of Cavalieri's principle in 2-dimension.

Proper reflections with vertical axis and translations to the horizontal direction can match the areas, denoted by $f$, $g$, $h$ and $i$, to the left side of the line $\mathrm{LM}$. By collecting their reflected and translated images, the rectangle $PQML$ can be constructed.

## References


[1] Problems 1761–1770, *Parabola* **61**(1), 2025.

[2] J.R. Voelkel, *Johannes Kepler and the New Astronomy*, Oxford University Press, 1999.

[3] Wikipedia, *Orthographic projection*, last accessed on 2026-08-31.

[4] Wikipedia, *Ellipse*, last accessed on 2026-08-31.

[5] Wikipedia, *Cavalieri's principle*, last accessed on 2026-08-31.

[6] Wikipedia, *$e$ (mathematical constant)*, last accessed on 2026-08-31.

[7] Wikipedia, *Cylinder*, last accessed on 2026-08-31.

[8] Wikipedia, *Kepler's laws of planetary motion*, last accessed on 2026-08-31.

[9] Wikipedia, *Astronomical unit*, last accessed on 2026-08-31.