Parabola Volume 62 Issue 2 (2026), Article 7

One-dimensional subdivision rules

Anaya Kale[1] and **Brian Rushton**[2]

1 Introduction

Subdivision rules, which are ways of creating fractals, have been studied extensively in dimension two, and have been more recently studied in other dimensions as well. One-dimensional subdivision rules have the simplest structure, but they have not received sufficient attention. In this paper, we provide definitions, examples, and fundamental results on combinatorial classification of these one-dimensional subdivision rules, focusing on those with a small number of tile types.

2 Background

In two dimensions, subdivision rules are used to generate fractals, and have applications to geometric group theory [1] and the study of rational maps [2]. Informally, a two-dimensional subdivision rule is a rule for taking a labelled tiling of a two-dimensional space and generating a sequence of more and more refined labelled tilings that are nested (so that each closed tile is contained completely in a closed tile from the previous tiling in the sequence) and locally defined (so that any two tiles with the same label will have subdivided the same way in the next tiling in the sequence).

Famous examples of subdivision rules include barycentric subdivision, where the only shapes are triangles and every triangle is subdivided into six smaller triangles, and binary subdivision, where every shape is a square and every square is subdivided into four other squares. [1, 5]

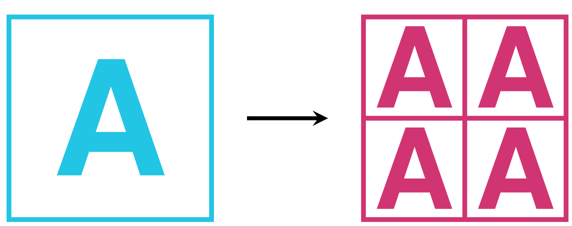

Figure 1: Two-dimensional binary subdivision

[1]Anaya Kale is a sophomore student at The Westwood School, Dallas.
[2]Brian Rushton is a math teacher at The Westwood School, Dallas.

They were later generalised to all dimensions; see [6]. In this paper, we are only interested in one-dimensional subdivision rules, which create sequences of tilings of the line. In particular, we will study only the combinatorial properties of the tilings.

Therefore, we make the following definitions.

Definition 1. *Given a set S with elements $\{A_1, A_2, \ldots, A_n\}$, let S^+ be the set of non-empty finite sequences made from elements in that set.*

For instance, if S has a single element A, then

$$S^+ = \{A, AA, AAA, AAAA, \ldots\}.$$

Definition 2. *A one-dimensional combinatorial subdivision rule R consists of the following:*

(a) A finite set

$$S_R = \{A_1, \ldots, A_n\}$$

of symbols called the tile types *of R.*

(b) A mapping

$$R : S_R \to S_R^+$$

called the subdivision mapping, *sending each tile type to a finite sequence of tile types.*

In the rest of this paper, we will use the word "subdivision rule" to mean a one-dimensional combinatorial subdivision rule, unless otherwise indicated.

The function from S_R to S_R gives us a function from

$$R : S_R^+ \to S_R^+.$$

This is because every element in S_R^+ is made up of elements of S_R, so we just apply the function to each part. For more rigorous details, see [4].

An R-complex X is any element of S_R^+. The mapping R sends X to a new element $R(X)$, which is called a subdivision of X. Continuing, we can get a sequence of subdivisions

$$X,\ R(X),\ R^2(X), \ldots.$$

These subdivisions can be interpreted graphically by using a line to represent each symbol.

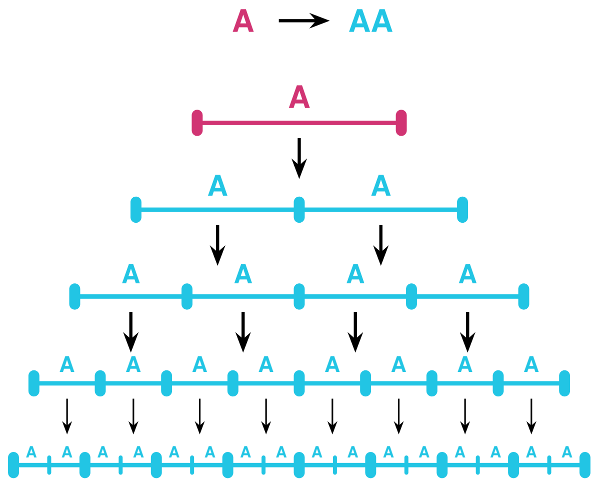

Figure 2: One dimensional binary subdivision

In the above picture, the subdivision complex is just the set $\{A\}$ and the subdivision mapping sends A to AA. Thus,

$$X = A\,,$$
$$R(X) = AA\,,$$
$$R^2(X) = AAAA\,,$$

and so on.

Another example is the Middle Thirds subdivision rule used in the Cantor set.[3] One way to represent it is with a subdivision complex that has two tiles $\{A, B\}$, where subdivision sends A to ABA and B to B.

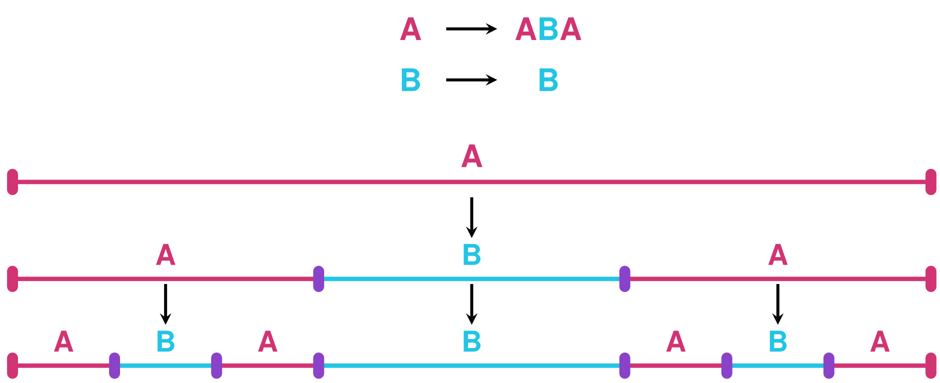

Figure 3: The Middle Thirds subdivision rule.

Starting with a single A tile, we get

$$X = A\,,$$
$$R(X) = ABA\,,$$
$$R^2(X) = ABABABA\,,$$

and so forth.

3 Properties of one-dimensional subdivision rules

3.1 Subdivision rules with a single tile type

The simplest type of subdivision rules are those with a single tile type. One example is binary subdivision, shown earlier. Below is another example, the trivial subdivision rule with a single tile that does not divide.

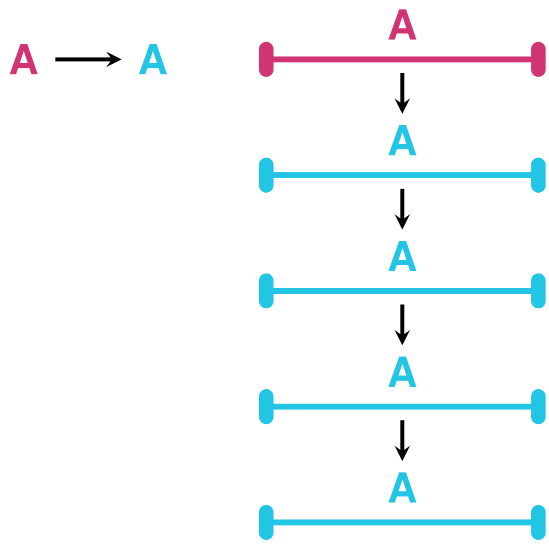

Figure 4: The trivial subdivision rule

These subdivision rules with a single tile type can be classified completely as follows.

Theorem 3. *The subdivision rules with one tile type correspond exactly with the set of positive whole numbers.*

Proof. First, to prove existence, let n be any natural number. Let Y be the subdivision rule where A is mapped to n copies of A. This is the desired subdivision rule.

To prove uniqueness, suppose that Y is a subdivision rule with one tile type A, where A subdivides into n copies of A. Since there is only one tile type, the order does not matter, so the only possibility is

$$\underbrace{AA\cdots A}_{n\text{ times}}.$$

Therefore, there is a unique subdivision rule for each positive whole number n. □

3.2 Subdivision rules with two tile types

When there are two tile types, such as A and B, the structure of subdivision rules becomes more complex. Each tile type may subdivide into a combination of both types, making interactions between them. As a result, the growth of the system depends on the interaction between the two tile types rather than just a single repeating rule. Unlike with just one tile type, two tile types can produce more complex design sequences.

Example

Consider the subdivision rule R with tile types A and B and the rules

$$A \to AB, \qquad B \to A\,.$$

That is, each A tile produces both an A and a B tile, while B only makes an A tile.

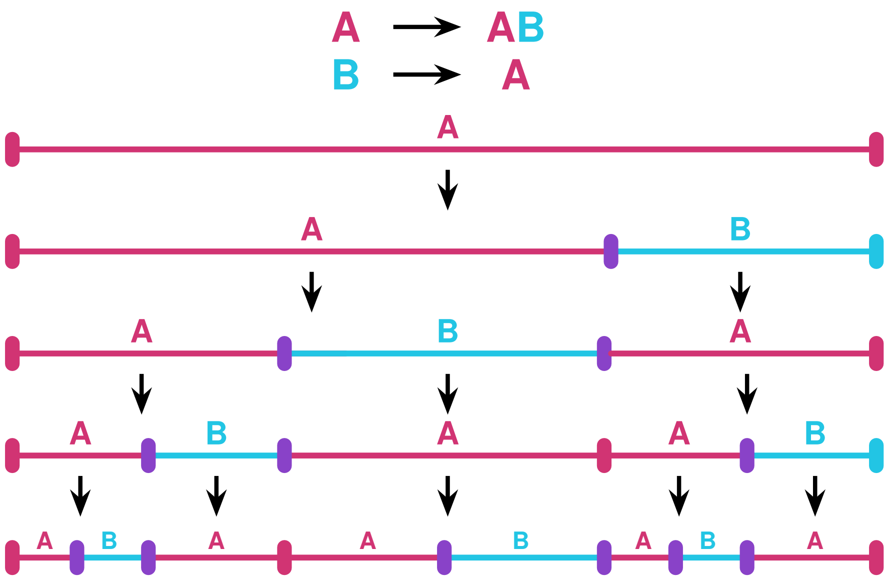

Figure 5: The Fibonacci subdivision rule, starting from a single A.

Starting with the R-complex X consisting of a single A tile, we obtain the following sequence:

$$\begin{aligned} X &= A\,, \\ R(X) &= AB\,, \\ R^2(X) &= ABAAB\,, \\ R^3(X) &= ABAABABA\,, \end{aligned}$$

and so forth. If we let $A(n)$ represent the number of A tiles in $R^n(X)$, then we can see from the above that

$$A(0) = 1\,, \quad A(1) = 1\,, \quad A(2) = 2\,, \quad A(3) = 3\,, \quad A(4) = 5\,,$$

which matches the beginning of the Fibonacci sequence.

This occurs because every A tile produces one A tile and one B tile, and every B tile produces only one A tile. As the subdivision is repeated, the number of A tiles at each stage becomes equal to the sum of the numbers from the two prior stages, making it follow the Fibonacci sequence.

Theorem 4. *$A(n)$ is equal to the nth term in the Fibonacci sequence.*

Proof. The proof is by induction. Let $F(n)$ denote the nth Fibonacci number, and let $B(n)$ be the number of B tiles in $R^n(X)$.

Base cases.
When $n = 0$, the number of A tiles is $A(0) = 1 = F(0)$, and the number of B tiles is 0.
When $n = 1$, $A(1) = 1 = F(1)$.

Induction step.
Assume that $A(n) = F(n)$ and $B(n) = F(n-1)$.
Since each A tile is subdivided into AB and each B tile is subdivided into A,

$$A(n+1) = A(n) + B(n) = F(n) + F(n-1) = F(n+1)\,,$$

and

$$B(n+1) = F(n)\,.$$

Thus, by induction, the statement is true for all n. □

3.3 Equivalent subdivision rules

Finally, we discuss equivalent subdivision rules. Sometimes, different subdivision rules can produce the same overall behaviour or growth. Even if the patterns look different, then the number of tiles of each type can grow in the same way.

For example, consider the following two subdivision rules:

(a) The set $\{A, B\}$ with the rules

$$A \to AB\,, \qquad B \to A\,.$$

(b) The set $\{A, B\}$ with the rules

$$A \to BA\,, \qquad B \to A\,.$$

These rules are different in order, but they produce the same number of A and B tiles at each step. This is because each A tile produces one A and one B, and each B tile creates one A. The order of the tiles does not matter for the overall count because rearranging the order does not change how many or what types of tiles are produced.

This shows that the arrangement of tiles does not matter, but rather how many of each type are created. Thus, two subdivision rules can be considered equivalent if they produce the same number of tiles of each type at every step, even if their patterns are different. Therefore, the arrangement of tiles does not determine growth; instead, the number of tiles produced by each tile type determines the growth.

4 Conclusion

In this paper, we studied one-dimensional subdivision rules with one tile type. We showed that one-dimensional subdivision rules with one tile type are determined by the number of tiles into which each tile subdivides, and because of this there is exactly one subdivision rule for each positive whole number.

We then studied one-dimensional subdivision rules with two tile types. We found that these rules can create more complicated patterns; in particular, we showed that the subdivision rule

$$A \to AB\,, \qquad B \to A$$

generates the Fibonacci sequence. This demonstrates that even simple subdivision rules can create complicated mathematical patterns through repeated subdivision.

Finally, we observed that even when the order of the subdivision tiles is different, the rules can have the same growth behaviour. This shows that the growth of a subdivision rule depends on what and how many tiles each tile type produces, rather than on the order of the subdivision tiles.

References

[1] J.W. Cannon, W.J. Floyd and W.R. Parry, Finite subdivision rules, *Conformal Geometry and Dynamics* **5** (2001), 153–196.

[2] J.W. Cannon, W.J. Floyd and W.R. Parry, Constructing subdivision rules from rational maps, *Conformal Geometry and Dynamics* **11** (2007), 128–136.

[3] J. R. Munkres, *Topology*, 2nd ed., Prentice Hall, Upper Saddle River, NJ, 2000.

[4] L. Rubin and B. Rushton, Enumeration of m-endomorphisms, *Involve* **9** (2016), 423–438.

[5] B. Rushton, *Subdivision Rules, 3-Manifolds, and Circle Packings*, Ph.D. dissertation, Brigham Young University, Provo, UT, 2012.

[6] B. Rushton, A finite subdivision rule for the n-dimensional torus, *Geometriae Dedicata* **167** (2013), 355–370.