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Mathematical formulation of the basic arch structures of Gothic architecture

Songjia Ren[1]

1 Introduction

Gothic architecture originated in France in the 12th century. It constantly evolved over the centuries. Gothic architecture combines structural engineering with mathematical aesthetics. It features pointed arches, ribbed vaults and flying buttresses, representing a complex combination of geometric principles and religious symbols. Like Notre-Dame de Rouen and St. Owen's Church in Rouen, they are proofs of the profound understanding of mathematics and physics by medieval builders. The builders transformed these understandings of mathematics and physics into the beautiful order expressed in stones.

Gothic architecture involves algebraic relations. The tangent circles and intersecting geometric patterns achieved structural efficiency and are aesthetically pleasing to viewers. From a mathematical perspective, the pointed arch can be described as the intersection point of two circles, and the function of these structures in Gothic architectures could be found by calculations. When the pointed arch ascends towards the sky, it also creates an optimal force distribution. Similarly, the ribbed arch top that can cover the huge internal space follows an exact geometric pattern and can effectively transfer the load to the support columns and buttresses. This article explores how these mathematical concepts were applied to late Gothic churches.

The significance of this research is not limited to these aspects of historical architecture and culture. Through this research, we can understand the mathematics used by medieval builders and how they transformed abstract principles into bricks and stones to create architectural structures that could inspire people's spirit and intelligence. This can provide some experience for the contemporary era in terms of design and structural efficiency. The research also proves how mathematical thinking builds a bridge between artistic expression and engineering. The architectural structures it creates are both beautiful and practical. This paper studies specific cases of Gothic architecture and conducts a detailed analysis of its patterns. It aims to reveal the mathematical language that endows humanity with some of the most inspiring sacred spatial forms.

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2 Assumptions

Assumption 1: I assume that all models are in an ideal environment that features neither weather, wind, erosion nor gravitational forces.
Justification 1: The model is based on an ideal situation in which we only focus on the mathematical expression of the structures and patterns, so it will not adjust the different environmental situations of different churches. That adjustment is attainable but is too complex and deviates from this model's focus.

Assumption 2: I assume that all materials used have the same physical properties.
Justification 2: This simplifies our calculations, and it is unfeasible for our model to contain all data from all the materials in use.

3 Problems

Recent research has interpreted Gothic architecture through geometric and structural patterns in supporting the aesthetic nature. Hales [5] shows that Gothic structures can be described using geometric construction methods, focusing on the influence of ratio on architecture forms.

Bork [2] highlights that the pointed arch was widely used in Gothic architecture due to its structural properties to redirect loads more vertically and reduce lateral thrust in tall stone buildings. Similarly, Zenner [8] emphasises that medieval architects used implicit geometric reasoning to achieve optimised aesthetic presentations without formal mathematical notations.

These studies highlight a geometric and structural interpretation of Gothic architecture. Few works explicitly formulate algebraic models of individual components, although some studies discuss the geometric constructions of structural elements in Gothic construction, such as the construction of pentagonal elements geometrically [**?**, **?**]. This motivates the mathematical modelling approach in this paper.

When studying Gothic Churches, basic structures such as pointed arches and trefoils are the basic unit of research. Considering these basic structures helps create a deeper understanding of the complex flamboyant and radiant patterns involved in Gothic churches. Following the motivation of algebraic models, the key question is: What are the algebraic expressions of arches in Gothic churches?

4 Arches

Gothic Architecture is characterised by a series of basic structures and patterns. Classic structures are pointed arches and flying buttresses. Algebraically, I analysed the functions and geometric shapes that formed the arches.

Unlike the semicircular arches of Romanesque architecture, pointed arches have an upward tapering point. Pointed arches are formed by two intersecting circular arcs. This design can be mathematically described using algebraic curves. The two circles intersect at the top point and have the same radius with the same relative distance to the centre of the arch. This is shown in Figure 1.

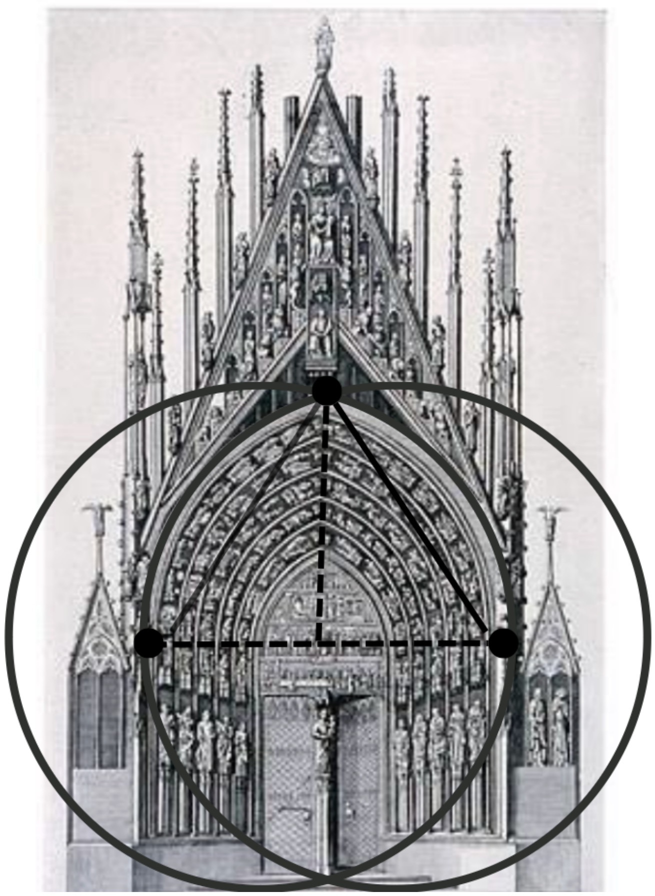

Figure 1: Pointed arch, “Sculptures médiévales de la Cathédrale de Strasbourg”, Ministère de la Culture, 2022 [7]

5 The 2D geometrical analysis

We will now set up the framework to define a two-dimensional arch as superimposed in Figure 1. This arch is isolated and presented in Figure 2.

The vertical height of the arch is h.
Two circles intersect at the vertex of the arch $(0, h)$.
The width of base of the arch is $2a$.
The centres of the two circles are $(\pm d, 0)$.
The circles have radius R.

The radius R is the variable that we are changing to affect the taper of the arch. The width of the base is kept constant. Hence, the distance of the circle centre to the arch centre will change as the taper of the arch changes.

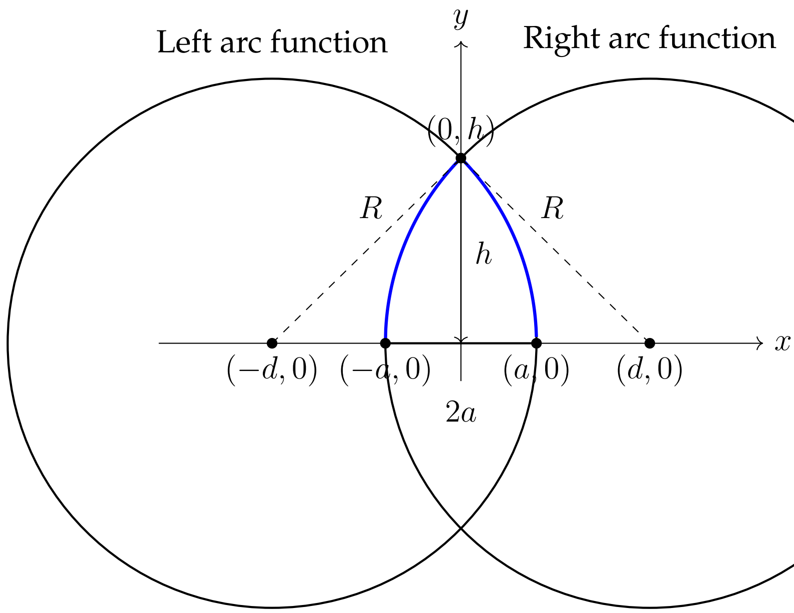

Figure 2: Arc dimensions

The left circle function is defined by $(x+d)^2 + y^2 = R^2$.
Similarly, the right circle function is given by $(x-d)^2 + y^2 = R^2$.
The distance of the centre of the circles, $(\pm d, 0)$ (the ends of the base of the two arcs), to the vertex $(0, h)$ is

$$d^2 + h^2 = R^2 \,. \tag{1}$$

Note that we can also write

$$d = R - a \,. \tag{2}$$

We can then rewrite (1) using (2) to solve for h in terms of a and R:

$$h = \sqrt{R^2 - d^2} = \sqrt{R^2 - (R-a)^2} = \sqrt{2aR - a^2} \,.$$

This gives the information needed for constructing the arcs:

Left arc function: $(x+d)^2 + y^2 = R^2$, where $x \geq 0$ and $0 \leq y \leq \sqrt{2aR - a^2}$;
Right arc function: $(x-d)^2 + y^2 = R^2$, where $x \leq 0$ and $0 \leq y \leq \sqrt{2aR - a^2}$.

These are the generation functions for the pointed arch, where we can change the radius and the width of the base as needed. Using these equations, we can generate a pointed arch for $a = 1.5$, $R = 2.25, 3, 4.5$ as shown in Figure 3.

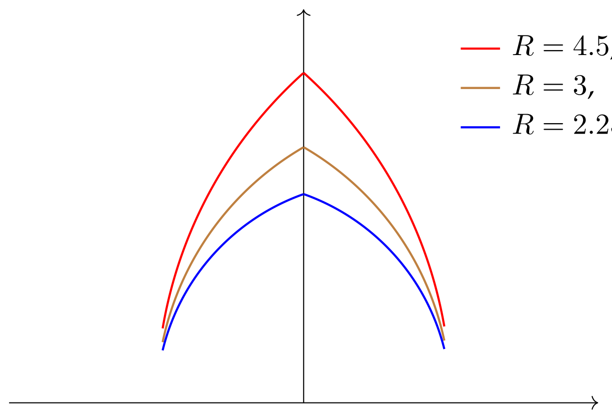

Figure 3: Generation function of pointed arch, $a = 1.5$, $R = 2.25, 3, 4.5$

6 3D promotion

Note that in real life, the arches are not just simple circles but are formed by three-dimensional materials. This means that we need to promote the model from 2D to 3D. Gothic churches usually have archways which have a slant tilting inwards. The shape we need to form an archway can be thought of as a cube with an elliptical paraboloid being dug out. This paraboloid is then defined by

$$\frac{(x-a)^2 + z^2}{R} < -y\,, \tag{3}$$

where R controls the radius of the arch and a controls the centre of the arch. We need

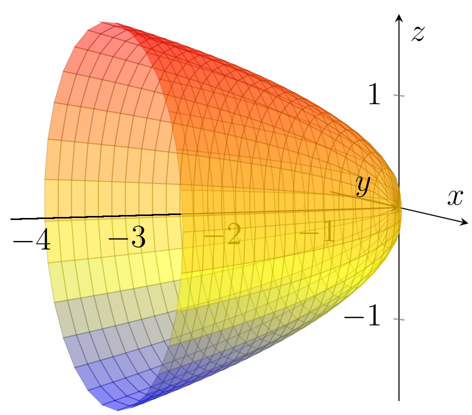

Figure 4: Elliptical paraboloid defined by $(x-a)^2 + z^2 = -Ry$ with $a = 0$, $R = 1$.

the opposite of the paraboloid so the inequality is flipped. To define the archway, we need separate functions for the left and right arch, similar to the 2D set up. The functions and constraints for the archway are as follows:

$$\begin{array}{lll} \text{Left arc:} & (x-a)^2 + z^2 \geq -Ry\,, & \{-5 < y < -4\,,\ -6 < x < 0\,,\ 0 < z < 6\}\,; \\ \text{Right arc:} & (x+a)^2 + z^2 \geq -Ry\,, & \{-5 < y < -4\,,\ \ 0 < x < 6\,,\ 0 < z < 6\}\,. \end{array}$$

This forms a generation function for arches in the form of three-dimensional, as shown in Figure 5.

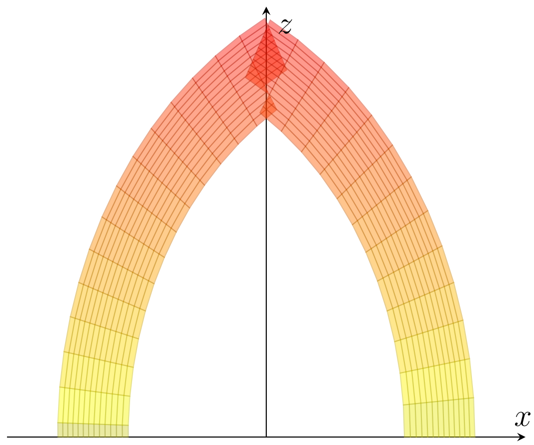

Figure 5: 3D arch

7 Classical arches

The taper of an arch is measured by the ratio of the width of the arch to the radius of the two arcs. There are four commonly used forms of arches in the Gothic Buildings, shown in Figure 6.

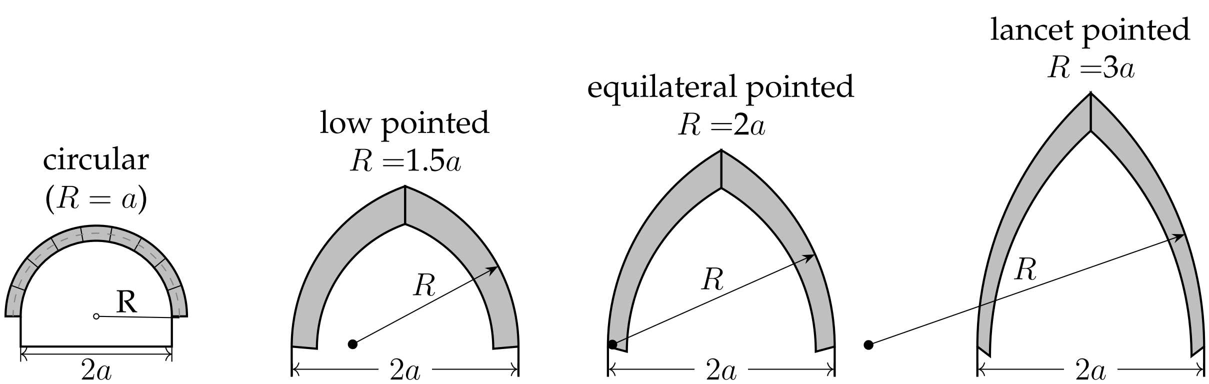

Figure 6: Arch types

7.1 Circular arch

The circular arch is a distinctive symbol of Romanesque architecture. It is characterised by $R = a$, shown in Figure 6. It has a relatively large lateral thrust. It was commonly used in ancient Rome architectures and bridges because of its stability, as seen in the Pont du Gard, Figure 7.

Figure 7: Pont du Gard, Roman Aqueduct, photograph by Benh Lieu Song [1].

Figure 8: Rouen Cathedral West Facade, photograph supplied by author.

7.2 Low pointed arch

The low pointed arch is a little higher than the circular arches. It is characterised by $R = \frac{3a}{2}$. This arch is typically used in buildings with a certain height but with moderate pointedness. For example in Figure 8, the Rouen Cathedral has used this kind of arch in its front window's arch decorations.

7.3 Equilateral pointed arch

The equilateral pointed arch is characterised by $R = 2a$. Equilateral pointed arches are widely used in Gothic architecture because they improve vertical load transfer and reduce lateral thrust in tall structures. The doorways and windows in the Cologne Cathedral have used equilateral pointed arches frequently, as shown in Figure 9.

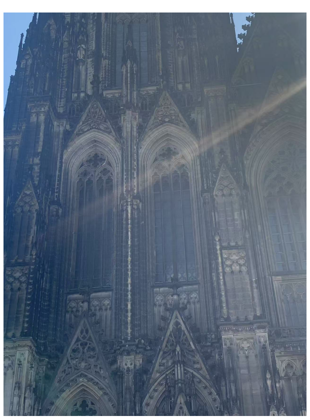

Figure 9: Cologne Cathedral, photograph supplied by author.

Figure 10: Saint Maclou in Rouen, photograph supplied by author.

7.4 Lancet pointed arch

The lancet pointed arch is $R = 3a$. It is commonly found in the late Gothic architectures. It can effectively transfer force to the ground and have a relatively low lateral thrust. The church of Saint Maclou in Rouen uses this kind arch in its side doorways as seen in Figure 10.

8 Decorative arches

Arches in Gothic Churches are not only used for the purpose of mechanic use. It is also used for decorations in the churches. They became a canvas for the artistic expression. The decorative potential of arches evolved significantly during the Gothic period. It is blending geometric precision with mechanics purposes [2]. Below is a mathematical formulation of key decorative arch types and functions can generate them.

8.1 Ogee arch

The Ogee arch is an example of arches created for decorative purposes. The Ogee opening is weak structurally for major load-bearing openings [3]. This weaker structural function implies this kind of arch served for more aesthetic purposes. The arch is formed by two parts: two circles on the top intersecting to form the tip and two circles on the bottom to form the arch, as shown in Figure 11. From this geometric condition, we could create a piecewise function to generate the Ogee arch. I am using two variables R and r each representing the radius of the circular arc on the top and bottom. This way, we could freely alter the shape and ratio of the Ogee arch in the function. First model the upper part, which is two circles intersecting with each other. We have

$$\begin{aligned}(x-R)^2+y^2&=R^2, \quad y\leq 0, \quad x\leq \frac{R}{\sqrt{2}};\\(x+R)^2+y^2&=R^2, \quad y\leq 0, \quad x\geq -\frac{R}{\sqrt{2}}.\end{aligned}$$

The restrictions to x are due to where it is attached to the two circles at the bottom. Now consider the two circles at the bottom connected with the upper circles. Present the function using a different constant r to represent another radius such that

$$\begin{aligned}\left(x-\frac{R}{2}\right)^2+(y-h)^2&=r^2, \quad y\geq h, \quad x\geq \frac{R}{\sqrt{2}};\\\left(x+\frac{R}{2}\right)^2+(y-h)^2&=r^2, \quad y\geq h, \quad x\leq -\frac{R}{\sqrt{2}}.\end{aligned}$$

These two circles are displaced from the centre by $\frac{R}{2}$ horizontally and h vertically to connect to the upper circles. Here, h is the constant

$$h=-\sqrt{(R+r)^2-\left(\frac{R}{2}\right)^2}$$

shown in Figure 12.

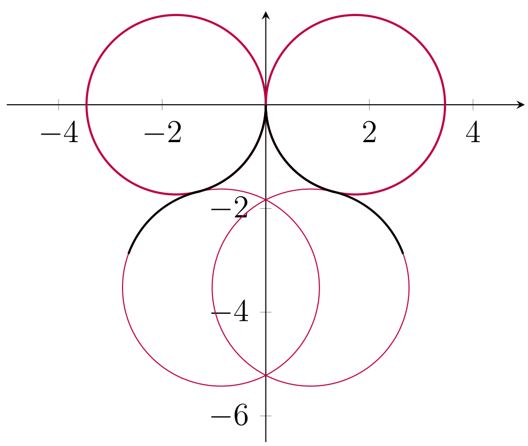

Figure 11: Ogee arch

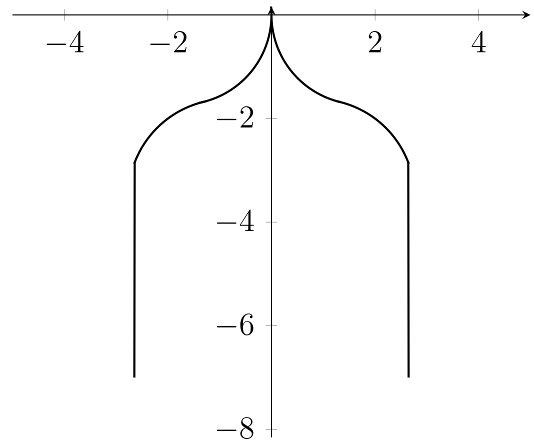

Figure 12: Ogee arch

8.2 Trefoil pointed arch

In Gothic architecture, the pointed trefoil arch was widely used for portals inside the church for decorative purposes. It is often interpreted as symbolising the Holy Trinity in Christian cultures [4]. Other than its religious symbols, it is also aesthetically geometric. The trefoil pointed arch, shown in Figure 12, is formed by two parts: two circles on the top intersecting to form the arch tip and two circles on the bottom to form another arch. From this geometric condition, we could also create a piecewise function to generate this kind of arch. We use two variables r and a, where r controls the radius of the circular arcs and a controls the distance of the circular arches is away from the arch centre. First, we construct the bottom arches with two circular arcs and construct the upper arch based on the variables of the base arch. This way, we could freely alter the shape and ratio of the trefoil pointed arch in this function, shown in Figure 14.

First, two circular arcs from the circles $(x-a)^2+y^2=r^2$ and $(x+a)^2+y^2=r^2)$ are taken, restricted respectively to the domains $(x \geq a\,,\ y \geq 0)$ and $(x \leq -a\,,\ y \geq 0)$. Next, two upper arcs are taken from circles $(x-a)^2+(y-r)^2=4a^2$ and $(x+a)^2+(y-r)^2=4a^2$, restricted respectively to the regions $(x \leq 0)$ and $(x \geq 0)$.

8.3 Trefoil round arch

This kind of arch is similar to the trefoil pointed arch. The difference between the trefoil pointed arch and the trefoil round arch is in its name: the top will now be rounded. It is also often interpreted as symbolising the Holy Trinity in the Christian cultures [4]. This affects its geometric properties so now there will be only one half circular arc overlapping the apex of the arch, shown in Figure 15. We could get the piecewise function through its geometric property. Here, we are also using two variables to control the shape and its form, shown in Figure 16.

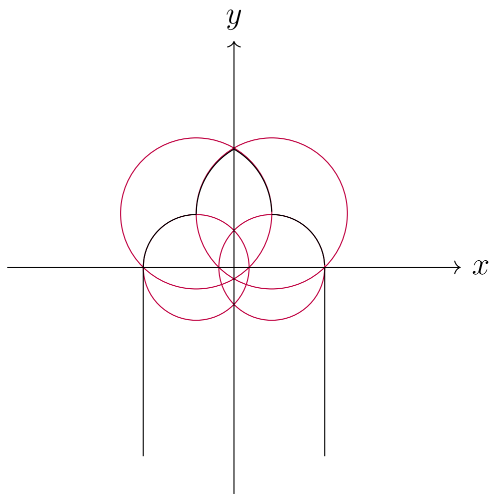

Figure 13: Trefoil pointed arch

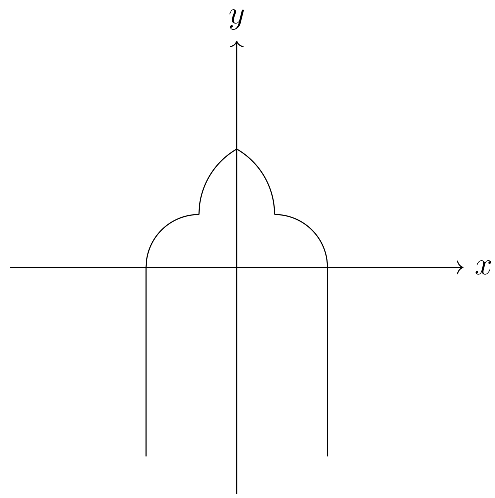

Figure 14: Trefoil pointed arch

First, two arcs are taken from the circles $(x-a)^2+y^2=r^2$ and $(x+a)^2+y^2=r^2$. These arcs are restricted to the domains $(x \geq a\,,\ y \geq 0)$ and $(x \leq -a\,,\ y \geq 0)$, respectively, forming the two symmetric side arcs. Next, the upper part of the curve is obtained from the circle $x^2+(y-r)^2=r^2$, restricted to the region $(y \geq r)$, which forms the central top arc.

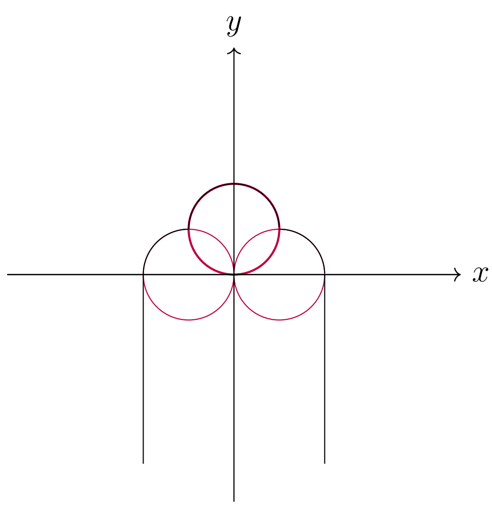

Figure 15: Trefoil round arch

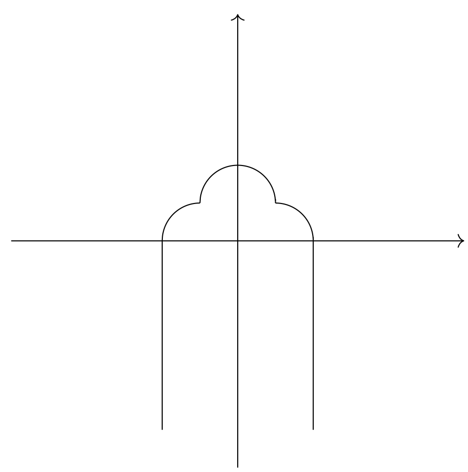

Figure 16: Trefoil round arch

8.4 Multifoil round arch

The multifoil round arch is an architectural feature defined by a semicircular outer profile. It is formed by identical circles tangent to each other and all inscribed in a larger circle. It allows for any even number of inset circles. It is formed by a set of different circle functions, given by the following properties:

The centres of the small circular arcs are uniformly distributed on the large circle.
There are n circles distributed in the large circle, where n is even.
The angles between neighbouring circle centres is $\frac{2\pi}{n}$.
The arch is obtained by considering the upper semi-circle of the resulting pattern.
All circular arcs are inscribed in a large circle with radius R.
The smaller circular arcs have radius r, where $r < R$.

The distance between the centres of any two neighbouring smaller circles is $2r$ because they are tangent to each other. There is also another way of expressing this distance by treating this distance as a chord of the circle with radius $R - r$, which we can express as $2(R - r)\sin\left(\frac{\pi}{n}\right)$. So, we have the equation

$$2(R - r)\sin\left(\frac{\pi}{n}\right) = 2r\,,$$

and we can therefore write r as

$$r = \frac{R\,\sin\left(\frac{\pi}{n}\right)}{1 + \sin\left(\frac{\pi}{n}\right)}\,. \tag{4}$$

The next thing that we are going to establish is the position of the centre of the small circular arcs and the point of their tangent to each other.

Every centre of these smaller circles are on the circle with radius $R - r$. So, the k^{th} circle has the centre C_k at $\left((R-r)\cos\left(\frac{2k\pi}{n}\right),\ (R-r)\sin\left(\frac{2k\pi}{n}\right)\right)$, where $k = 0, 1, \ldots, n-1$. Using (4), we find that

$$R - r = \frac{R\big(1 + \sin\left(\frac{\pi}{n}\right)\big) - R\sin\left(\frac{\pi}{n}\right)}{1 + \sin\left(\frac{\pi}{n}\right)} = \frac{R}{1 + \sin\left(\frac{\pi}{n}\right)}\,. \tag{5}$$

Hence by substitutions,

$$C_k = \left(\frac{R\cos\left(\frac{2k\pi}{n}\right)}{1 + \sin\left(\frac{\pi}{n}\right)}\,,\ \frac{R\sin\left(\frac{2k\pi}{n}\right)}{1 + \sin\left(\frac{\pi}{n}\right)}\right)\,. \tag{6}$$

With the coordinates for the centre and the radius $R-r$, we can now obtain the function for the k^{th} circle. It is

$$\left(x - \frac{R\cos\left(\frac{2k\pi}{n}\right)}{1 + \sin\left(\frac{\pi}{n}\right)}\right)^2 + \left(y - \frac{R\sin\left(\frac{2k\pi}{n}\right)}{1 + \sin\left(\frac{\pi}{n}\right)}\right)^2 = \left(\frac{R\sin\left(\frac{\pi}{n}\right)}{1 + \sin\left(\frac{\pi}{n}\right)}\right)^2\,. \tag{7}$$

Now we calculate the point at which circles are tangent to each other. It is the midpoint of the line connecting their centres. The midpoint of the centres C_k and C_{k+1} is

$$T_k = \frac{1}{2}\frac{R}{1 + \sin\left(\frac{\pi}{n}\right)}\left(\cos\left(\frac{2k\pi}{n}\right) + \cos\left(\frac{(2k+2)\pi}{n}\right),\ \sin\left(\frac{2k\pi}{n}\right) + \sin\left(\frac{(2k+2)\pi}{n}\right)\right)\,.$$

Simplify using sum-to-product trigonometric identities, this give us

$$T_k = \frac{R\cos\left(\frac{\pi}{n}\right)}{1+\sin\left(\frac{\pi}{n}\right)}\left(\cos\left(\frac{(2k+1)\pi}{n}\right), \sin\left(\frac{(2k+1)\pi}{n}\right)\right). \tag{8}$$

Connecting tangent points to get a line function would restrict the circular arcs above it to form the arch shape. The linear function is

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}.$$

Substitute in T_k, T_{k+1} from (8) to get

$$\begin{aligned} x_1 &= \frac{R\cos\left(\frac{\pi}{n}\right)}{1+\sin\left(\frac{\pi}{n}\right)}\cos\left(\frac{2k\pi+\pi}{n}\right) & y_1 &= \frac{R\cos\left(\frac{\pi}{n}\right)}{1+\sin\left(\frac{\pi}{n}\right)}\sin\left(\frac{2k\pi+\pi}{n}\right) \\ x_2 &= \frac{R\cos\left(\frac{\pi}{n}\right)}{1+\sin\left(\frac{\pi}{n}\right)}\cos\left(\frac{2k\pi+3\pi}{n}\right) & y_2 &= \frac{R\cos\left(\frac{\pi}{n}\right)}{1+\sin\left(\frac{\pi}{n}\right)}\sin\left(\frac{2k\pi+3\pi}{n}\right). \end{aligned}$$

The gradient is

$$m = \frac{y_2-y_1}{x_2-x_1} = \frac{\sin\left(\frac{(2k+3)\pi}{n}\right)-\sin\left(\frac{(2k+1)\pi}{n}\right)}{\cos\left(\frac{(2k+3)\pi}{n}\right)-\cos\left(\frac{(2k+1)\pi}{n}\right)} = -\cot\left(\frac{2(k+1)\pi}{n}\right).$$

Therefore, the line function is

$$\frac{y-(R-r)\cos\left(\frac{\pi}{n}\right)\sin\left(\frac{(2k+1)\pi}{n}\right)}{x-(R-r)\cos\left(\frac{\pi}{n}\right)\cos\left(\frac{(2k+1)\pi}{n}\right)} = -\cot\left(\frac{2(k+1)\pi}{n}\right).$$

Using (4), rearrange to get

$$y = \frac{R\cos^2\left(\frac{\pi}{n}\right)}{\sin\left(\frac{2(k+1)\pi}{n}\right)\left(1+\sin\left(\frac{\pi}{n}\right)\right)} - x\cot\left(\frac{2(k+1)\pi}{n}\right). \tag{9}$$

Now we have all the functions needed to construct this pattern. We first have a set number k where $k = 0, 1, \ldots, n-1$. There is a function corresponding to each k. From (7), we have the function

$$\left(x - \frac{R\cos\left(\frac{2k\pi}{n}\right)}{1+\sin\left(\frac{\pi}{n}\right)}\right)^2 + \left(y - \frac{R\sin\left(\frac{2k\pi}{n}\right)}{1+\sin\left(\frac{\pi}{n}\right)}\right)^2 = \left(\frac{R\sin\left(\frac{\pi}{n}\right)}{1+\sin\left(\frac{\pi}{n}\right)}\right)^2. \tag{10}$$

Then we use (9) to gain the conditions for the domain and range.

When the centre of the circle C_k is above the x-axis, we need the upper half of the function, which is

$$y \geq \frac{R\cos^2\left(\frac{\pi}{n}\right)}{\sin\left(\frac{2(k+1)\pi}{n}\right)\left(1+\sin\left(\frac{\pi}{n}\right)\right)} - x\cot\left(\frac{2(k+1)\pi}{n}\right). \tag{11}$$

When the centre of the circle C_k is on the x-axis, the range needed become a vertically partitioning the function, which we restrict with the x coordinate of the centre of the circle:

$$x \leq -\frac{R\cos\left(\frac{\pi}{n}\right)}{1+\sin\left(\frac{\pi}{n}\right)}\cos\left(\frac{(2k+1)\pi}{n}\right). \tag{12}$$

When the centre of the circle C_k is below the x-axis, we need the lower half of the function, which is

$$y \leq \frac{R\cos^2\left(\frac{\pi}{n}\right)}{\sin\left(\frac{2(k+1+\frac{n}{2})\pi}{n}\right)\left(1+\sin\left(\frac{\pi}{n}\right)\right)} - x\cot\left(\frac{2\left(k+1+\frac{n}{2}\right)\pi}{n}\right). \tag{13}$$

The arch that we need is just the parts above the x-axis, shown in Figure 17. Just adding the constraint of every circle needing to have $y \geq 0$ could give us the multifoil round arch. The reason why the initial function was defined to include the circles below the x-axis is because it would be useful in constructing other patterns in the flamboyant and rayonnant style patterns.

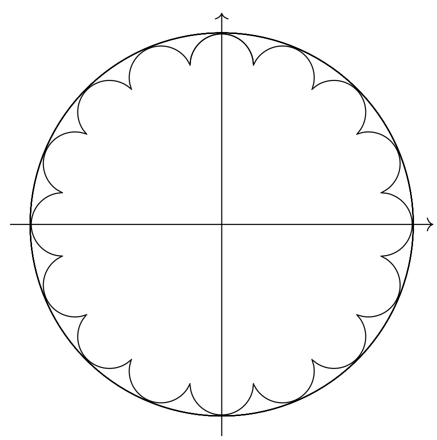

Figure 17: The whole multifoil shape with $n = 16$

8.5 Cinquefoil round arch

This is a special case for the multifoil round arch, where we take the value of $n = 8$ in (10)–(13). This arch formed by five circular arches is notable for its religious significance. The number five has tremendous symbolism in Medieval Christian culture. It can symbolise the five wounds of Jesus and the Five Joys of the Virgin Mary [3]. The angle of the placement of the circular arcs is calculated and creates a generative function of the Cinquefoil Arch, as shown in Figure 18.

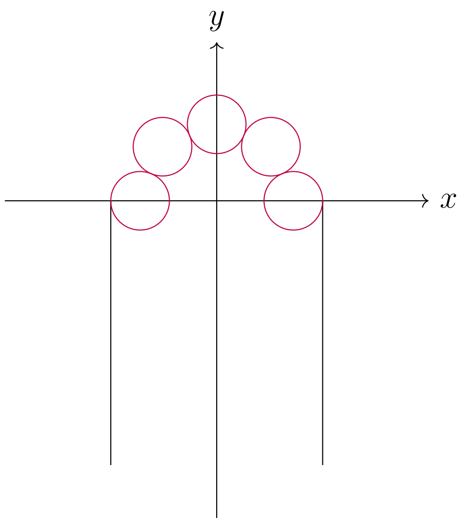

Figure 18: Cinquefoil round arch

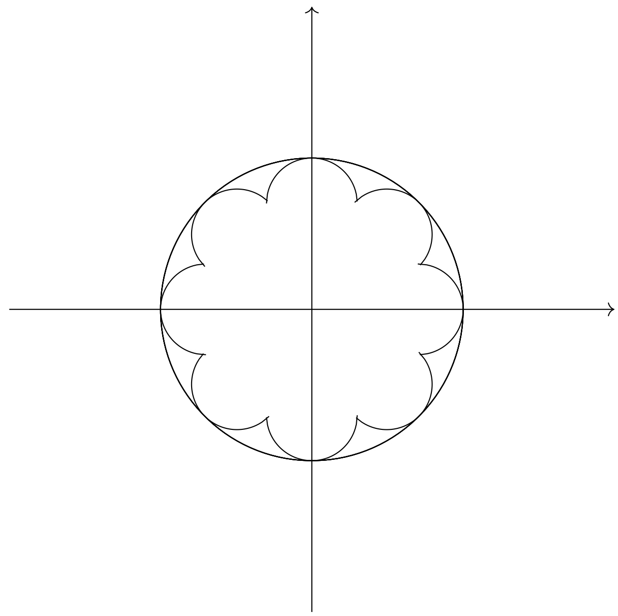

Figure 19: Full cinquefoil circle

9 Conclusion

This paper examined the mathematical principles of decorative Gothic arches by using algebraic and geometric methods. It demonstrated how complex foil-based decorations can be generated from precise mathematical relationships. The analysis shows that Gothic architecture is both visually expressive and mathematically systematic. Although the models are idealized, they provide insight into how medieval buildings contain abstract mathematical ideas within architectural structures. Overall, this study highlights mathematics as a unifying language between engineering and artistic expression, revealing how Gothic architecture achieves its distinctive balance of beauty and structural logic.

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