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64th Annual UNSW School Mathematics Competition: Competition problems and solutions

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A Junior Division – Problems

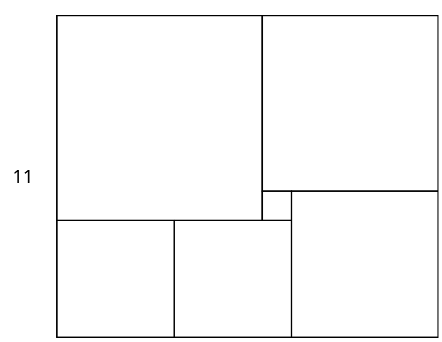
Problem A1:

In a school mathematics competition, 23 students scored either 10, 11, 12, or 13 marks. The total number of marks scored by all the students is 253. It is known that the number of students who scored 12 marks is one and a half times the number of students who scored 13 marks.

How many students scored 12 marks?

Problem A2:

A rectangle is partitioned into several squares, as shown in the figure below. The length of the shorter side of the rectangle is 11. Find the length of the longer side of the rectangle.



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Problem A3:

Use Fermat's Little Theorem[2] to prove that $7^{120} - 1$ is divisible by 143.

Problem A4:

Let $N \geq 2$ be an integer. Given N integers, prove that there exist two whose difference is divisible by $N - 1$.

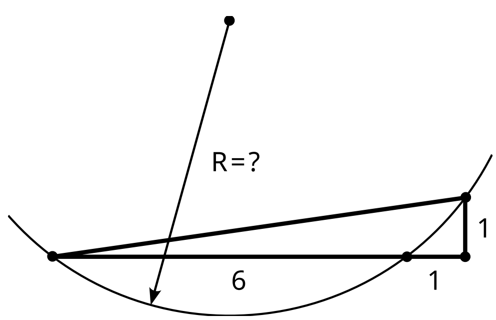
Problem A5:

Two players play the following game. The game begins with the number 2026 on the board. On each turn, a player subtracts from the current number a positive power of 3 that does not exceed it (including $3^0 = 1$).[3] The players alternate turns, and the player who makes the number 0 wins.

Determine which player has a winning strategy, and describe a strategy that guarantees victory regardless of the other player's moves.

Problem A6:

An arc of a circle passes through the endpoints of the hypotenuse of a right triangle. The arc intersects one of the legs of the triangle, dividing it into two segments of lengths 1 and 6, as shown in the figure below. If the length of the other leg is 1, find the radius of the circle.



[2]

Theorem 1 (Fermat's Little Theorem)**.** *Let p be a prime and let A be an integer not divisible by p. Then*

$$A^{p-1} \equiv 1 \pmod{p}.$$

The notation $x \equiv a \pmod{p}$ means that $p \mid (x - a)$; that is, the difference $x - a$ is divisible by p.

[3]That is, numbers of the form 3^k where k is an integer satisfying $k \geq 0$.

B Senior Division – Problems

Problem B1:

Two positive real numbers a and b satisfy

$$\frac{a}{64+a^2} = \frac{ab}{64+(ab)^2} = \frac{b}{64+b^2}.$$

Determine the set of all possible values of $a+b$.

Problem B2:

Let a_1, a_2, a_3 and b_1, b_2, b_3 be six numbers between -1 and 1. Prove that

$$|a_1a_2a_3 - b_1b_2b_3| \leq |a_1 - b_1| + |a_2 - b_2| + |a_3 - b_3|.$$

Problem B3:

Two players play the following game. The game begins with the number 2026 on the board. On each turn, a player subtracts from the current number a positive power of 3 that does not exceed it (including $3^0 = 1$).[4] The players alternate turns, and the player who makes the number 0 wins.

Determine which player has a winning strategy, and describe a strategy that guarantees victory regardless of the other player's moves.

Problem B4:

Find the sum:

$$\lfloor\sqrt{1}\rfloor + \lfloor\sqrt{2}\rfloor + \cdots + \lfloor\sqrt{2025}\rfloor.$$

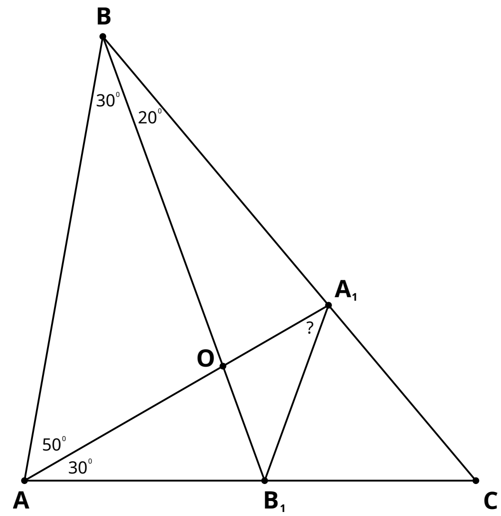
The function $\lfloor x \rfloor$ gives the greatest integer less than or equal to x. For example, $\lfloor 1 \rfloor = 1$, $\lfloor \frac{3}{2} \rfloor = 1$, $\lfloor \frac{199}{100} \rfloor = 1$, and $\lfloor \frac{201}{100} \rfloor = 2$.

Problem B5:

Alice and Bob are locked in two separate rooms with big red buttons. In the next hour, they must each press their button exactly once, and their presses must occur within 10 minutes of each other to escape. Assuming they press at random, what is the probability they escape?

[4] That is, numbers of the form 3^k where k is an integer satisfying $k \geq 0$.

Problem B6:
Find $\angle AA_1B_1$ in the diagram below.



A Junior Division – Solutions

Solution A1.

Note that there is also the trivial solution in which all 23 students scored 11 marks, giving $a = c = d = 0$ and $b = 23$. The argument below determines the non-trivial solution.

Let a, b, c, and d be the numbers of students who scored 10, 11, 12, and 13 marks, respectively. We are given that the number of students who scored 12 marks is one and a half times those who scored 13 marks, so

$$c = \frac{3}{2}d.$$

In particular, d must be even. The total number of students is

$$a + b + c + d = 23,$$

and the total number of marks is

$$10a + 11b + 12c + 13d = 253.$$

Substituting $c = \frac{3}{2}d$ into the marks equation gives

$$10a + 11b + 12\left(\frac{3}{2}d\right) + 13d = 253,$$

which simplifies to

$$10a + 11b + 18d + 13d = 253 \quad \Longrightarrow \quad 10a + 11b + 31d = 253.$$

Since $10a + 11b \geq 0$, we must have

$$31d \leq 253 \quad \Longrightarrow \quad d \leq 8.$$

Because d is even, the possible values are $d = 2, 4, 6, 8$.

Case analysis:

- $d = 8$:
$$10a + 11b = 253 - 31 \cdot 8 = 5,$$
which has no nonnegative integer solution.

- $d = 6$:
$$10a + 11b = 253 - 31 \cdot 6 = 67,$$
which has no nonnegative integer solution.

- $d = 4$:
$$10a + 11b = 253 - 31 \cdot 4 = 129.$$
A solution is $a = 3$, $b = 9$, but then
$$a + b + c + d = 3 + 9 + 6 + 4 = 22 \neq 23,$$
so this case is invalid.

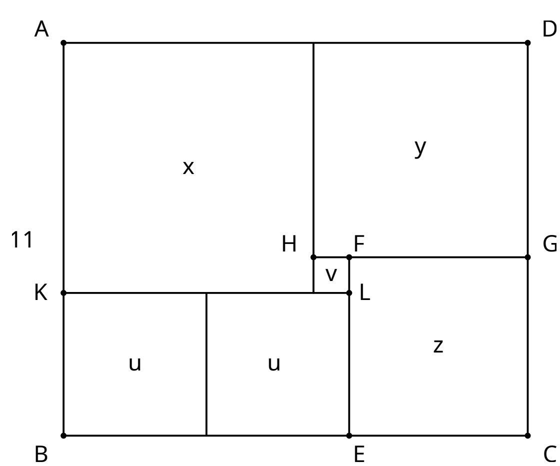
- $d = 2$:
$$10a + 11b = 253 - 31 \cdot 2 = 191.$$
Possible solutions are $(a, b) = (18, 1)$ and $(7, 11)$.
Now check the total number of students:
$$a + b + c + d = a + b + 3 + 2 = a + b + 5 = 23 \quad \Longrightarrow \quad a + b = 18.$$
Only $(a, b) = (7, 11)$ satisfies this condition.

Thus, $d = 2$ and
$$c = \frac{3}{2} \cdot 2 = 3.$$

Answer: $\boxed{3}$ □

Solution A2.

We solve this problem by setting up and solving a system of linear equations. As in the figure below, label the squares from largest to smallest by
$$x,\ y,\ z,\ u,\ v\ ,$$
where each variable denotes the side length of the corresponding square.



Reading off equal lengths along the indicated segments gives the following equations:

$$\begin{array}{ll} \text{(AB)} & x+u=11\,, \\ \text{(CD)} & y+z=11\,, \\ \text{(EF)} & u+v=z\,, \\ \text{(GH)} & z+v=y\,, \\ \text{(KL)} & x+v=2u\,. \end{array}$$

So we solve the system

$$\begin{cases} x+u=11\,, \\ y+z=11\,, \\ u+v=z\,, \\ z+v=y\,, \\ x+v=2u\,. \end{cases}$$

From $u+v=z$ we have $z=u+v$. Substituting into $z+v=y$ gives

$$y=(u+v)+v=u+2v\,.$$

Now substitute $y=u+2v$ and $z=u+v$ into $y+z=11$:

$$(u+2v)+(u+v)=11 \quad\Longrightarrow\quad 2u+3v=11\,.$$

From $x+u=11$ we have $x=11-u$. Substitute this into $x+v=2u$:

$$(11-u)+v=2u \quad\Longrightarrow\quad 11+v=3u \quad\Longrightarrow\quad u=\frac{11+v}{3}.$$

Plug into $2u+3v=11$:

$$2\cdot\frac{11+v}{3}+3v=11 \quad\Longrightarrow\quad 22+11v=33 \quad\Longrightarrow\quad v=1.$$

Hence

$$u=\frac{11+1}{3}=4\,, \qquad z=u+v=5\,, \qquad y=u+2v=6\,, \qquad x=11-u=7\,.$$

The rectangle's longer side is $x+y$, so

$$x+y=7+6=13\,.$$

Therefore, the longer side of the rectangle equals $\boxed{13}$. $\square$

Solution A3.

Since $143 = 11 \cdot 13$, it suffices to show that $7^{120} - 1$ is divisible by both 11 and 13.

First consider modulo 11. Since 11 is prime and 7 is not divisible by 11, Fermat's Little Theorem gives

$$7^{10} \equiv 1 \pmod{11}.$$

Raising both sides to the 12th power,

$$(7^{10})^{12} = 7^{120} \equiv 1^{12} \equiv 1 \pmod{11}.$$

Thus $7^{120} - 1$ is divisible by 11.

Now consider modulo 13. Since 13 is prime and 7 is not divisible by 13, Fermat's Little Theorem gives

$$7^{12} \equiv 1 \pmod{13}.$$

Raising both sides to the 10th power,

$$(7^{12})^{10} = 7^{120} \equiv 1^{10} \equiv 1 \pmod{13}.$$

Thus $7^{120} - 1$ is divisible by 13.

Since $7^{120} - 1$ is divisible by both 11 and 13, and these primes are relatively prime, it follows that $7^{120} - 1$ is divisible by 143.[5] $\square$

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[5]

Proof of Fermat's Little Theorem. For completeness, we include a proof of Fermat's Little Theorem.

Let p be prime and suppose A is not divisible by p. Consider the $p - 1$ numbers

$$A, 2A, 3A, \ldots, (p-1)A.$$

We claim that no two of these are congruent modulo p. Suppose that

$$aA \equiv bA \pmod{p}.$$

Then $(a - b)A \equiv 0 \pmod{p}$. Since A is not divisible by p, it follows that

$$a - b \equiv 0 \pmod{p}.$$

Thus $a \equiv b \pmod{p}$. Because $1 \leq a, b \leq p - 1$, we must have $a = b$.

Therefore the numbers $A, 2A, \ldots, (p-1)A$ are distinct modulo p, and hence are a rearrangement of the nonzero residues $1, 2, \ldots, p - 1$. Multiplying them together gives

$$A \cdot 2A \cdot \cdots \cdot (p-1)A \equiv 1 \cdot 2 \cdot \cdots \cdot (p-1) \pmod{p}.$$

Thus

$$(p-1)!\, A^{p-1} \equiv (p-1)! \pmod{p}.$$

Since $(p-1)!$ is not divisible by p, we may cancel it modulo p, obtaining

$$A^{p-1} \equiv 1 \pmod{p}.$$

$\square$

Solution A4.

Let the integers be $I_1, I_2, \ldots, I_N$. Consider their remainders when divided by $N-1$. Since division by $N-1$ yields exactly $N-1$ possible remainders,

$$0, 1, 2, \ldots, N-2,$$

there are only $N-1$ possible remainder classes modulo $N-1$. Because we have N integers but only $N-1$ possible remainders, the Pigeonhole Principle implies that at least two of the integers must have the same remainder when divided by $N-1$.

Suppose

$$I_i = a(N-1) + r \quad \text{and} \quad I_j = b(N-1) + r,$$

for some integers a, b and some remainder r. Then their difference is

$$I_i - I_j = (a-b)(N-1),$$

which is divisible by $N-1$. Therefore, there exist two of the given integers whose difference is divisible by $N-1$. □

Solution A5.

The *second player* has a winning strategy. The key idea is for the second player to always ensure that, after their move, the number on the board is an even integer greater than 2.

Initially, the game starts at 2026, which is even and greater than 2, so this condition already holds before the first player's move.

On each turn, the first player subtracts a power of 3, and every power of 3 is odd. Therefore, after the first player's move, the number on the board becomes odd. If this odd number is 1 or 3, then the second player wins immediately by subtracting 1 or 3, respectively. If the odd number is greater than 3, then the second player subtracts 1, returning the number on the board to an even number greater than 2.

By continuing in this way, the second player always restores the game to an even number greater than 2. Eventually, the first player must face the number 4. From 4, the first player has only two legal moves:

$$4 \to 3 \qquad \text{or} \qquad 4 \to 1.$$

In either case, the second player wins immediately:

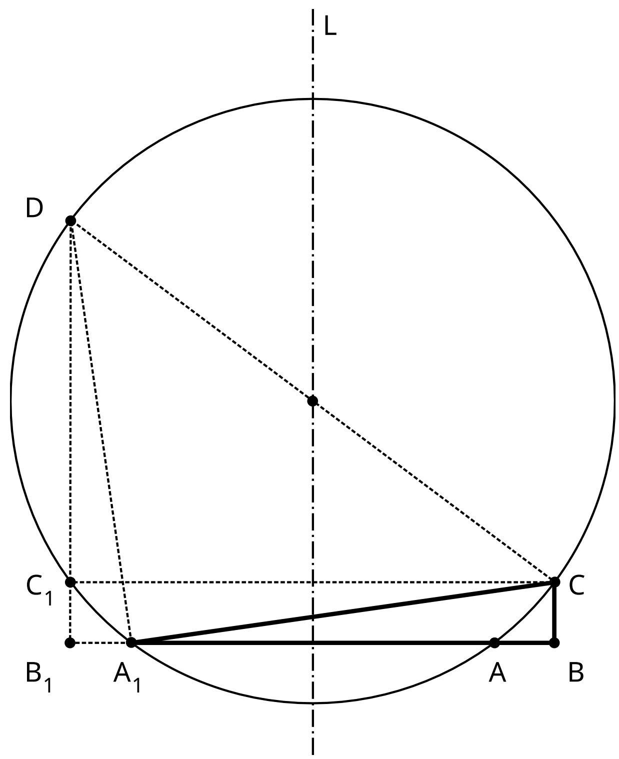
$$3 \to 0 \quad \text{by subtracting } 3, \qquad 1 \to 0 \quad \text{by subtracting } 1.$$

Thus, the second player can always force a win. □

Solution A6.

Let us denote the right triangle by $\triangle A_1BC$, where A_1C is the hypotenuse. Let A be the point where the circle intersects the leg A_1B.

Let L be the line through the center of the circle perpendicular to A_1B. Reflect points B and C across the line L to obtain points B_1 and C_1, respectively. Let D be the point on the circle diametrically opposite to C. Connect the relevant points as shown in the figure.



From the given data,

$$AB = 1, \quad A_1A = 6, \quad BC = 1, \quad A_1B_1 = 1.$$

Let

$$DC_1 = x,$$

where x is unknown.

We use two facts: (1) the Pythagorean Theorem, and (2) an angle subtended by a diameter of a circle is a right angle. Since DC is a diameter, triangle $\triangle DC_1C$ is right-angled, and triangle $\triangle DA_1C$ is also right-angled.

From triangle $\triangle DC_1C$,

$$DC^2 = DC_1^2 + C_1C^2.$$

Now

$$C_1C = A_1B + B_1A_1 = 7 + 1 = 8,$$

so

$$DC^2 = x^2 + 8^2 = x^2 + 64.$$

From triangle $\triangle DA_1C$,

$$DC^2 = DA_1^2 + A_1C^2.$$

First compute A_1C^2 from right triangle $\triangle A_1CB$:

$$A_1C^2 = A_1B^2 + BC^2 = 7^2 + 1^2 = 49 + 1 = 50.$$

Next compute DA_1^2 from right triangle $\triangle DB_1A_1$:

$$DA_1^2 = DB_1^2 + B_1A_1^2.$$

Since $DB_1 = DC_1 + C_1B_1 = x + 1$, we obtain

$$DA_1^2 = (x+1)^2 + 1^2.$$

Thus,

$$DC^2 = (x+1)^2 + 1 + 50.$$

Equating the two expressions for DC^2,

$$x^2 + 64 = (x+1)^2 + 51.$$

Expanding,

$$x^2 + 64 = x^2 + 2x + 1 + 51,$$

so

$$x^2 + 64 = x^2 + 2x + 52.$$

Canceling x^2 gives

$$64 = 2x + 52,$$

hence

$$2x = 12 \quad \Rightarrow \quad x = 6.$$

Therefore,

$$DC^2 = 6^2 + 64 = 36 + 64 = 100,$$

so

$$DC = 10.$$

Since DC is the diameter, the radius is

$$\boxed{5}.$$

□

B Senior Division – Solutions

Solution B1.

Note that there is also the trivial solution in which all 23 students scored 11 marks, giving $a = c = d = 0$ and $b = 23$. The argument below determines the non-trivial solution.

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Now check the total number of students:
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Only $(a, b) = (7, 11)$ satisfies this condition.

Thus, $d = 2$ and
$$c = \frac{3}{2} \cdot 2 = 3.$$

Answer: $\boxed{3}$ □

Solution B2.

Consider the difference $a_1a_2a_3 - b_1b_2b_3$. Add and subtract the terms $a_1a_2b_3$ and $a_1b_2b_3$:
$$\begin{aligned} a_1a_2a_3 - b_1b_2b_3 &= a_1a_2a_3 - a_1a_2b_3 + a_1a_2b_3 - a_1b_2b_3 + a_1b_2b_3 - b_1b_2b_3 \\ &= a_1a_2(a_3 - b_3) + a_1b_3(a_2 - b_2) + b_2b_3(a_1 - b_1). \end{aligned}$$

Taking absolute values and applying the triangle inequality yields
$$|a_1a_2a_3 - b_1b_2b_3| \leq |a_1a_2||a_3 - b_3| + |a_1b_3||a_2 - b_2| + |b_2b_3||a_1 - b_1|.$$

Since $|a_j| \leq 1$ and $|b_j| \leq 1$ for $j = 1, 2, 3$, we have
$$|a_1a_2| \leq 1, \qquad |a_1b_3| \leq 1, \qquad |b_2b_3| \leq 1.$$

Therefore,
$$|a_1a_2a_3 - b_1b_2b_3| \leq |a_3 - b_3| + |a_2 - b_2| + |a_1 - b_1|,$$
which proves the desired inequality. □

Solution B3.

The *second player* has a winning strategy. The key idea is for the second player to always ensure that, after their move, the number on the board is an even integer greater than 2.

Initially, the game starts at 2026, which is even and greater than 2, so this condition already holds before the first player's move.

On each turn, the first player subtracts a power of 3, and every power of 3 is odd. Therefore, after the first player's move, the number on the board becomes odd. If this odd number is 1 or 3, then the second player wins immediately by subtracting 1 or 3, respectively. If the odd number is greater than 3, then the second player subtracts 1, returning the number on the board to an even number greater than 2.

By continuing in this way, the second player always restores the game to an even number greater than 2. Eventually, the first player must face the number 4. From 4, the first player has only two legal moves:

$$4 \to 3 \qquad \text{or} \qquad 4 \to 1.$$

In either case, the second player wins immediately:

$$3 \to 0 \quad \text{by subtracting } 3, \qquad 1 \to 0 \quad \text{by subtracting } 1.$$

Thus, the second player can always force a win. $\square$

Solution B4.

To find the sum $S = \sum_{n=1}^{2025} \lfloor \sqrt{n} \rfloor$, consider:

(a) For integers $k \geq 1$, $\lfloor \sqrt{n} \rfloor = k$ when n is in $[k^2, (k+1)^2 - 1]$.

(b) The interval $[k^2, (k+1)^2 - 1]$ contains $(k+1)^2 - k^2 = 2k + 1$ integers.

(c) Since $2025 = 45^2$, $\lfloor \sqrt{2025} \rfloor = 45$ occurs only once.

Thus, the sum is:

$$S = \sum_{k=1}^{44} k \cdot (2k+1) + 45 \cdot 1.$$

Expanding the summation:

$$S = \sum_{k=1}^{44} (2k^2 + k) + 45.$$

Using the formulas for sums:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

For $n = 44$:

$$\sum_{k=1}^{44} k^2 = \frac{44 \cdot 45 \cdot 89}{6} = 29370, \quad \sum_{k=1}^{44} k = \frac{44 \cdot 45}{2} = 990.$$

So:

$$S = 2 \cdot 29370 + 990 + 45 = 58740 + 990 + 45 = 59775.$$

Therefore, the sum is:

$$\boxed{59775}$$

□

Solution B5.

This boils down to the problem where the times of the button presses X_1 and X_2 are uniformly distributed random variables, $\mathcal{U}[0,1]$, and finding

$$\begin{aligned}
\mathrm{P}\left\{|X_1 - X_2| \le \frac{1}{6}\right\} =&\mathrm{P}\left\{-\frac{1}{6} \le X_1 - X_2 \le \frac{1}{6}\right\} \\
=&\mathrm{P}\left\{X_2 - \frac{1}{6} \le X_1 \le X_2 + \frac{1}{6}\right\} \\
=&\mathrm{P}\left\{0 \le X_2 \le \frac{1}{6} \bigcap 0 \le X_1 \le X_2 + \frac{1}{6}\right\} \\
&+ \mathrm{P}\left\{\frac{1}{6} \le X_2 \le \frac{5}{6} \bigcap X_2 - \frac{1}{6} \le X_1 \le X_2 + \frac{1}{6}\right\} \\
&+ \mathrm{P}\left\{\frac{5}{6} \le X_2 \le 1 \bigcap X_2 - \frac{1}{6} \le X_1 \le 1\right\}
\end{aligned}$$

Then,

$$\begin{aligned}
\mathrm{P}\left\{|X_1 - X_2| \le \frac{1}{6}\right\} =& \int_0^{1/6} \int_0^{x_2+1/6} dx_1 dx_2 \\
&+ \int_{1/6}^{5/6} \int_{x_2-1/6}^{x_2+1/6} dx_1 dx_2 \\
&+ \int_{5/6}^{1} \int_{x_2-1/6}^{1} dx_1 dx_2 \\
=&\frac{11}{36}
\end{aligned}$$

□

Solution B6.

To find $\angle AA_1B_1$, we first show that $\triangle OA_1B_1$ is isosceles. If $\triangle OA_1B_1$ is isosceles with $OA_1 = OB_1$, then $\angle OA_1B_1 = \angle OB_1A_1$. By the Triangle Sum Theorem (TST) applied to $\triangle AOB$, $\angle AOB = 100^\circ$, so $\angle A_1OB_1 = 100^\circ$. Applying the TST to $\triangle OA_1B_1$, the sum of the angles is 180°, so:

$$2 \times \angle OA_1B_1 + 100^\circ = 180^\circ \implies \angle OA_1B_1 = 40^\circ.$$

Thus, $\angle AA_1B_1 = \angle OA_1B_1 = 40^\circ$, assuming $\triangle OA_1B_1$ is isosceles. We now prove that $\triangle OA_1B_1$ is isosceles by showing $OA_1 = OB_1$.

Let $BA_1 = a$, $OA_1 = x$, and $OB_1 = y$. Our goal is to prove $x = y$. First, apply the TST to $\triangle AA_1B$. We find $\angle AA_1B = 80^\circ$. Similarly, in $\triangle BA_1O$, $\angle BOA_1 = 80^\circ$. $\triangle BA_1O$ is isosceles since $\angle OBA_1 = \angle OAB_1 = 80^\circ$, so $BO = BA_1 = a$. Also, $\triangle AA_1B$ is isosceles since it has two angles of 50°, so $AA_1 = BA_1 = a$.

To prove $\triangle OA_1B_1$ is isosceles, we use the Law of Sines (LoS). In $\triangle BA_1O$:

$$\frac{x}{\sin 20^\circ} = \frac{a}{\sin 80^\circ}, \qquad \text{so} \qquad a = x\frac{\sin 80^\circ}{\sin 20^\circ}.$$

Next, apply the TST to $\triangle ABB_1$ to find $\angle AB_1O = 70^\circ$. Then, by the LoS in $\triangle AOB_1$:

$$\frac{y}{\sin 30^\circ} = \frac{a - x}{\sin 70^\circ}.$$

Substitute $a = x\frac{\sin 80^\circ}{\sin 20^\circ}$:

$$y = \frac{\sin 30^\circ}{\sin 70^\circ}\left(x\frac{\sin 80^\circ}{\sin 20^\circ} - x\right) = x\frac{\sin 30^\circ}{\sin 70^\circ}\left(\frac{\sin 80^\circ}{\sin 20^\circ} - 1\right).$$

To show $x = y$, we need the coefficient of x to equal 1:

$$\alpha = \frac{\sin 30^\circ}{\sin 70^\circ}\left(\frac{\sin 80^\circ}{\sin 20^\circ} - 1\right) = \frac{\sin 30^\circ}{\sin 70^\circ \sin 20^\circ}\left(\sin 80^\circ - \sin 20^\circ\right).$$

Using the trigonometric identity $\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$:

$$\sin 80^\circ - \sin 20^\circ = 2\cos\frac{80^\circ + 20^\circ}{2}\sin\frac{80^\circ - 20^\circ}{2} = 2\cos 50^\circ \sin 30^\circ = 2\sin 40^\circ \sin 30^\circ,$$

since $\cos 50^\circ = \sin(90^\circ - 50^\circ) = \sin 40^\circ$. Now, compute the denominator:

$$\sin 70^\circ \sin 20^\circ = \cos 20^\circ \sin 20^\circ = \frac{1}{2}\sin 40^\circ,$$

since $\sin 2\theta = 2\sin\theta\cos\theta$. Thus:

$$\alpha = \frac{\sin 30^\circ}{\sin 70^\circ \sin 20^\circ} \cdot 2\sin 40^\circ \sin 30^\circ = \frac{\sin 30^\circ \cdot 2\sin 40^\circ \sin 30^\circ}{\frac{1}{2}\sin 40^\circ} = \frac{2\sin^2 30^\circ}{\frac{1}{2}}.$$

Since $\sin 30^\circ = \frac{1}{2}$:

$$\alpha = 1.$$

Thus, $y = x \cdot \alpha = x$. Hence, $\triangle OA_1B_1$ is isosceles, and $\angle AA_1B_1 = 40^\circ$. $\square$