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Solutions 1791–1800

Q1791 Consider an "old–fashioned" 12–hour clock face having an hour hand, minute hand and second hand. Is there any time at which the three hands are equally spaced around the clock face? That is, any time at which each pair of hands is separated by $120°$? If so, then find all such times.

SOLUTION The configuration under consideration can never occur. We'll regard $120°$ as one unit of angle. With all hands starting together at 12:00:00, suppose that all are equally spaced after the hour hand has travelled through an angle of α units. The minute hand travels 12 times as fast as the hour hand, so that at this point it has travelled through an angle 12α. The second hand travels 60 times as fast as the minute hand, so it will have travelled through an angle 720α. Now suppose that the minute hand is one unit (that is, $120°$) clockwise from the hour hand, and the second hand is one unit anticlockwise from the hour hand. (The opposite case is very similar and will be left to the reader.) Then $12\alpha - \alpha$ is one unit plus some complete number of revolutions, that is, $1 + 3x$ for some integer x. So we have

$$11\alpha = 1 + 3x ;$$

and for similar reasons,

$$719\alpha = -1 + 3y$$

where y is also an integer. Eliminating α from these equations and rearranging, we obtain

$$3 \times 719x - 3 \times 11y = -730 .$$

Since x and y are integers this is impossible, because the left–hand side is an integer which is a multiple of 3, and the right–hand side is not. So it can never happen that all three hands are equally spaced.

Q1792 Let x be a real number with $0 < x < 1$, and let $p = 1 - x$ and $q = \sqrt{4x - 3x^2}$. Prove that $3^p + 3^q > 4$.

SOLUTION The Arithmetic Mean-Geometric Mean Inequality for four terms states that

$$\frac{a_1 + a_2 + a_3 + a_4}{4} \geq \bigl(a_1 a_2 a_3 a_4\bigr)^{1/4}$$

for positive real numbers a_1, a_2, a_3, a_4. Taking $a_1 = a_2 = a_3 = 3^{p-1}$ and $a_4 = 3^q$ shows that

$$3^p + 3^q = 3^{p-1} + 3^{p-1} + 3^{p-1} + 3^q \geq 4 \times 3^{(3p+q-3)/4} ;$$

while taking $a_1 = 3^p$ and $a_2 = a_3 = a_4 = 3^{q-1}$ yields

$$3^p + 3^q = 3^p + 3^{q-1} + 3^{q-1} + 3^{q-1} \geq 4 \times 3^{(p+3q-3)/4} .$$

Our result will follow if we can show that *either* of the exponents in our last two expressions is positive. We have

$$\begin{aligned} 3p+q-3>0 \quad &\Leftrightarrow \quad \sqrt{4x-3x^2}>3x \\ &\Leftrightarrow \quad 4x>12x^2 \quad \Leftrightarrow \quad x<\tfrac{1}{3}\,; \end{aligned}$$

and

$$\begin{aligned} p+3q-3>0 \quad &\Leftrightarrow \quad 3\sqrt{4x-3x^2}>x+2 \\ &\Leftrightarrow \quad 7x^2-8x+1<0 \\ &\Leftrightarrow \quad (x-1)(7x-1)<0 \quad \Leftrightarrow \quad x>\tfrac{1}{7}\,. \end{aligned}$$

Since $\frac{1}{7}<\frac{1}{3}$ one, possibly both, of these last two inequalities must be true, and so $3^p+3^q>4$ is true for all x between 0 and 1.

Q1793 Let $n\geq 3$. Three different numbers are chosen at random from $\{\,1,2,\ldots,n\,\}$; what is the probability that they are in arithmetic progression?

How does this change if we delete the word "different"? In other words, what is the probability if choosing the same number more than once is possible?

SOLUTION The number of possible choices is $C(n,3)$. Suppose that the chosen numbers are $a<b<c$. There is one possible value for b if $c-a$ is even, and none if $c-a$ is odd. So the number of successful triples is equal to the number of pairs of distinct numbers a,c in which both are even or both odd. We will need to consider n even and n odd separately.

- If $n=2m$, then we must choose two of m odd numbers or two of m even numbers. The number of choices is $C(m,2)+C(m,2)$.
- If $n=2m+1$, then we must choose two of $m+1$ odd numbers or two of m even numbers. The number of choices is $C(m+1,2)+C(m,2)$.

So the required probability for even n is

$$\frac{2C(m,2)}{C(n,3)}=\frac{6m(m-1)}{n(n-1)(n-2)}=\frac{3(2m)(2m-2)}{2n(n-1)(n-2)}=\frac{3}{2(n-1)}\,,$$

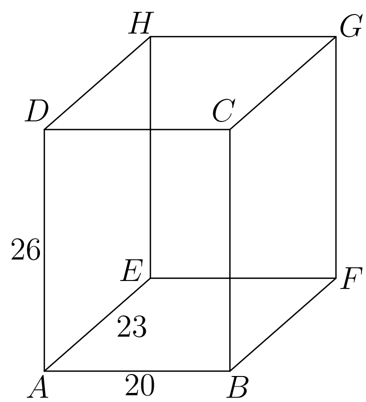
while for odd n it is

$$\frac{C(m+1,2)+C(m,2)}{C(n,3)}=\frac{3\big((m+1)m+m(m-1)\big)}{n(n-1)(n-2)}=\frac{3(2m)^2}{2n(n-1)(n-2)}=\frac{3(n-1)}{2n(n-2)}\,.$$

If choosing the same number repeatedly is allowed, then we can follow the same approach, with different counting methods. Alternatively, we can expand our set to $\{\,1,2,\ldots,n,L,U\,\}$, where L is interpreted as "repeat the lower number" and U as "repeat the upper number". For example, the choice $1,2,L$ would give the numbers $1,1,2$, while $1,2,U$ would give $1,2,2$ and $1,L,U$ would give $1,1,1$. So the overall number of choices of three numbers is as above, but with n replaced by $n+2$. To choose three numbers in arithmetic progression involves choosing two elements from even numbers and R, where R means "repeat"; or two elements from odd numbers and R. This again is the same as above but with n replaced by $n+2$. So the probability in this case is

$$\frac{3}{2(n+1)} \quad \text{if } n \text{ is even;} \qquad \frac{3(n+1)}{2(n+2)n} \quad \text{if } n \text{ is odd.}$$

Q1794 (a) Let $ABCDEFGH$ be a rectangular prism having side lengths $|AB| = 20$, $|AE| = 23$ and $|AD| = 26$. Taking AB, AE, AD to be the x, y, z axes respectively, a particle is projected from A along the line with equation $x = y = z$. There is a small aperture for the particle to escape at each of the vertices. Which vertex will the particle escape from?



(b) Can you find positive integers a, b, c to replace the dimensions $20, 23, 26$ in the previous part, in such a way that the particle exits from A?

SOLUTION When the particle exits through an aperture at a vertex, suppose that the total distance travelled in the AB direction is x. The particle must have travelled the whole width of the prism a complete number of times, and so x must be an integer which is a multiple of 20. The total distance travelled in the AE and AD directions must also be x, which is therefore also a multiple of 23 and of 26. These three conditions are equivalent to the single condition

$$x \quad \text{is a multiple of} \quad \mathrm{lcm}(20, 23, 26) = 20 \times 23 \times 13\ .$$

(Here lcm denotes the *least common multiple* of a set of integers.) The smallest positive x for which this is true is $x = 20 \times 23 \times 13$. This means that the particle has covered the whole span of the box 23×13 times in the AB direction, 20×13 times in the AE direction and 10×23 times in the AD direction. That is to say, an odd number of times in the AB direction and an even number in the other directions. Therefore the particle exits at B.

For the second question: no, there are no possible values of a, b, c for which the particle exits at A. To see this, follow through the previous argument to get

$$\mathrm{lcm}(a, b, c) = pa = qb = rc$$

where p, q, r are integers. If the particle exits at A, then p, q, r must all be even. But then $\mathrm{lcm}(a, b, c)/2$ is a multiple of all three numbers a, b, c, which contradicts what is meant by saying that $\mathrm{lcm}(a, b, c)$ is the **least** common multiple. So it is impossible for the particle to exit at A.

Q1795 We have a sequence of numbers $a_1, a_2, \ldots, a_n$. Each a_j is equal to 1 or 2, and the sum of all a_j is an even number $2A$. There is **no** sequence of consecutive numbers $a_p, a_{p+1}, \ldots, a_q$ with sum A. Find all possibilities for the original sequence.

SOLUTION Consider the first half of the sequence, including the middle term if n is odd: that is, the terms $a_1, \ldots, a_m$ where

$$m = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (n+1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

The sum of these terms, by assumption, is not equal to A; so it is either greater than A, or less than A. In the latter case, the sum of the *last* m terms is greater than A. So by reversing the sequence if necessary, we obtain a sequence which has all the given properties, and the additional property that the sum of the first m terms is $a_1 + \cdots + a_m > A$, where $2m$ is equal to n or $n + 1$.

Let $s_0, s_1, \ldots, s_n$ be the *partial sums* of the sequence: that is,

$$s_0 = 0\ , \quad s_1 = a_1\ , \quad s_2 = a_1 + a_2\ , \quad s_3 = a_1 + a_2 + a_3$$

and so on. Then $s_n = a_1 + a_2 + \cdots + a_n = 2A$. Note that for every $j \geq 1$ we have $s_j = s_{j-1} + a_j$, and therefore s_j equals $s_{j-1} + 1$ or $s_{j-1} + 2$. Moreover, we can write any sum of consecutive numbers from our sequence in terms of partial sums,

$$a_p + a_{p+1} + \cdots + a_q = s_q - s_{p-1}\ .$$

Since none of these expressions is equal to A, we have in particular that $s_j = s_j - s_0$ can never equal A. But the values of s_j increase from $s_0 < A$ to $s_m > A$, and so there must an index $k < m$ for which $s_k < A$ and $s_{k+1} > A$. Since consecutive values of s_j must differ by 1 or 2, the only way this can happen is that $s_k = A - 1$ and $s_{k+1} = A + 1$. Therefore, we know that the statement

$$s_j = 2j \quad \text{and} \quad s_{k+j} = A - 1 + 2j \tag{$*$}$$

is true when $j = 0$; we shall prove by mathematical induction that it is true for $j = 1, 2, \ldots, n - k$.

So, if $(*)$ is true for some specific j, then s_{k+j+1} is either $A+2j$ or $A+2j+1$. However, the former alternative is impossible as we should then have $s_{k+j+1} - s_j = A$; hence, $s_{k+j+1} = A + 2j + 1$. Now that we know this, we can use similar reasoning to say that $s_{j+1} \neq 2j + 1$, leaving as the only possible alternative $s_{j+1} = 2j + 2$. We have shown that $(*)$ is true when j is replaced by $j + 1$; so by induction, it is true for all j up to and including $j = n - k$.

We have formulae for $s_0, s_1, \ldots, s_{n-k}$, and for $s_k, s_{k+1}, \ldots, s_n$. Moreover,

$$2k < 2m \leq n+1 \quad \Rightarrow \quad n-k > k-1\ .$$

This means that there is no "gap" between the first group of s_j terms and the second, so we have formulae for all s_j. Moreover, the formula for s_n gives

$$2A = s_n = A - 1 + 2(n-k)\ ,$$

and this shows that A is an odd number. Therefore, $(*)$ shows that every s_j is even; hence every a_j is even, and our sequence consists of n terms, all equal to 2. Finally, this means that the sum of all a_j is $2n = 2A$, so $n = A$ is an odd number. Therefore, **answer**: the only sequences of the type under consideration are those which consist of an odd number of terms, all equal to 2.

Q1796 There is a pile of 2026 stones. Two players alternate in removing and throwing away stones so as to leave a smaller pile according to the rule: from n stones, one can leave either $n-1$ or $\lfloor n/2 \rfloor$ stones. Here $\lfloor n/2 \rfloor$ means $n/2$ rounded to the next integer downwards if n is odd: for example, from a pile of 123 stones the player can leave either 122 or 61. Whoever removes the last stone wins. If players always make the best possible moves, does the first or second player win?

SOLUTION Any positive integer n can be written in a unique way as $2^q r$, where $q \geq 0$ and r is odd. We shall say that a pile of $n = 2^q r$ stones has value $v(n) = q + r$.

We shall use mathematical induction to prove that if the game begins with n stones, then the first player wins if and only if $v(n)$ is odd. First of all, it is clear that this is true when $n = 1$: for in this case $v(n) = 0 + 1 = 1$ is an odd number, and the first player can certainly win by taking one (the last) stone.

Now suppose that there are n stones and that the result just stated is known to be true for all smaller numbers of stones. We consider various cases.

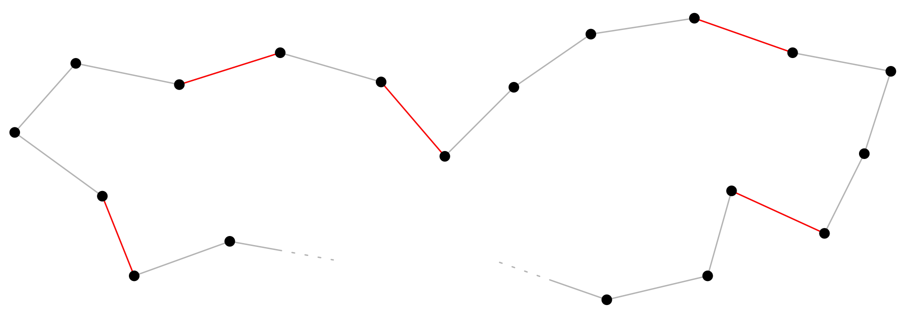
- Suppose that n is odd. Then $q = 0$, $r = n$ and $v(n) = n$ is an odd number. We have to prove that the first player can force a win. If we write $n = 2m + 1$, then the player's options are to leave $2m$ stones or $\lfloor (2m+1)/2 \rfloor = m$ stones. Since these numbers differ by a factor of 2, the values $v(m)$ and $v(2m)$ differ by 1, and so one of them must be even. Therefore, the player can choose to leave her opponent with an even–valued pile of stones, which is a certain loss for the other player and therefore a certain win for herself.
- Secondly, suppose that n is even, $n = 2^q r$ with $q \geq 1$, and that $v(n) = q + r$ is odd. We must show that the first player can win. In fact, this is done by using the "$n/2$" option. We have
$$v\Big(\Big\lfloor \frac{n}{2} \Big\rfloor\Big) = v\big(2^{q-1} r\big) = q - 1 + r$$
which is even; so the other player cannot win.

- Finally, suppose again that n is even, $n = 2^q r$ with $q \geq 1$, and this time suppose that $q + r$ is even. We have to prove that whichever move the player chooses, her opponent will be left with a winning position. If the first player leaves $\lfloor n/2 \rfloor = 2^{q-1}r$ stones, the remaining pile has value $q-1+r$ which is odd. If she leaves $n-1$ stones, then this is an odd number and the pile has value $n-1$. So either move by the first player leaves an odd–valued pile, a win for the other player, and the current player must lose.

In particular, we have $v(2026) = v(2^1 \times 1013) = 1014$, an even number, so in the case posed, the second player can (with accurate play) be certain of winning.

Q1797 There are a number of towns, some of which are connected by roads. Prove that, if there are $2n$ towns which are the endpoints of an odd number of roads, then one can make n separate trips which, between them, travel along all the roads exactly once each.

SOLUTION Split the "odd" towns into n pairs, and construct a new road between the towns in each pair. Now every town will be the endpoint of an even number of roads. According to a well–known fact about networks, this means that one can make a trip along all the roads, using each exactly once and returning to the start. Such a trip can be depicted schematically as follows.



In this diagram, towns may be shown more than once: that is, the dots need not all represent *different* towns. On the other hand, every road is shown once and only once. The newly constructed roads are coloured red. It is now clear that by removing the n red roads, we obtain n trips which, between them, travel along all the original roads exactly once each.

Comment. The "well–known fact" mentioned above actually only applies to *connected* networks. In the context of the present problem, this means that it is possible to travel by road from any town to any other. However, if this is not the case, we can treat the configuration as a collection of separate "islands", each with its own network of towns and roads, and our final conclusion is still true.

Q1798 Let $D = \{0, 1, 2, 3, 4\}$ be the set of base–5 digits. We wish to count sequences $d_1 d_2 \cdots d_n$ of n digits from D which satisfy the following condition:

$$\text{if } k = 1, 2, \ldots, n-1 \text{ and } d_k \neq 4, \text{ then } d_{k+1} > d_k.$$

That is, sequences in which every digit (other than the last) is succeeded by a larger digit, except that 4 may be succeeded by any digit. How many such sequences are there if $n = 10$?

SOLUTION Let $T(n)$ be the number of such sequences. Note that $T(0) = 1$ because the *empty string* (which has no digits at all!) satisfies our requirements. Suppose firstly that $n \geq 5$. Then every string we are counting contains a 4, since no such string of any length can contain more than four consecutive digits without a 4. Partition the sequences into those for which the last occurrence of 4 is d_n, $d_{n-1}, \ldots, d_{n-4}$. A string of length m ending in a 4 is obtained by taking any admissible string of length $m-1$ and appending a 4, so the number of such strings is $T(m-1)$. Therefore, there are $T(n-k)$ admissible strings $d_1 d_2 \cdots d_{n-k+1}$ ending with $d_{n-k+1} = 4$; to complete a string of n digits without any further 4s, we have to choose $k-1$ further digits from $\{0, 1, 2, 3\}$, which can be done in $C(4, k-1)$ ways, and then list them in increasing order (which can only be done in one way). Therefore. we have

$$\begin{aligned} T(n) &= \sum_{k=1}^{5} C(4, k-1) T(n-k) \\ &= T(n-1) + 4T(n-2) + 6T(n-3) + 4T(n-4) + T(n-5) \end{aligned}$$

for all integers $n \geq 5$.

In the case $n \leq 4$, we also have to count strings which do not contain any 4 at all, and there are $C(4, n)$ of these. So we have

$$T(n) = C(4, 0)T(n-1) + \cdots + C(4, n-1)T(0) + C(4, n)\ .$$

We can therefore compute successively

$$\begin{aligned} &T(0) = 1 \\ &T(1) = T(0) + C(4, 1) = 5 \\ &T(2) = T(1) + 4T(0) + C(4, 2) = 15 \\ &T(3) = T(2) + 4T(1) + 6T(0) + C(4, 3) = 45 \\ &T(4) = T(3) + 4T(2) + 6T(1) + 4T(0) + C(4, 4) = 140 \\ &T(5) = T(4) + 4T(3) + 6T(2) + 4T(1) + T(0) = 431 \\ &T(6) = T(5) + 4T(4) + 6T(3) + 4T(2) + T(1) = 1326 \end{aligned}$$

and so on, probably using computing assistance at some stage, until we obtain our answer $T(10) = 119305$.

Q1799 (a) Prove that, if the ratio of the side lengths in a rectangle is rational, then the rectangle can be cut into a collection of congruent isosceles triangles.

(b) We now explore the converse of (a). That is: if a rectangle can be cut into congruent isosceles triangles, then does it necessarily follow that the ratio of side lengths of the rectangle is rational? This will involve a number of questions over the next few issues of *Parabola*, so first we ask the following.

Prove that, if a rectangle can be cut into congruent isosceles **right–angled** triangles, then the ratio of side lengths of the rectangle is rational.

SOLUTION For (a), if a rectangle has side lengths a and b, where the ratio a/b is equal to a rational number m/n, then we have $a/m = b/n$. Call these (equal) quotients s; then $a = ms$ and $b = ns$, so the rectangle can be cut into mn squares of size $s \times s$. Then the squares can all be halved to get a collection of congruent isosceles triangles.

For (b), suppose that a rectangle has been cut into congruent isosceles right triangles. By scaling, we may assume that every triangle has side lengths 1, 1 and $\sqrt{2}$. Every point on the boundary must be covered by a point of one of the triangles, with no overlap. So each side of the rectangle must be made up of a certain number of sides of the triangles, and a certain number of hypotenuses. That is, the side lengths must be $a + b\sqrt{2}$ and $c + d\sqrt{2}$, where a, b, c, d are non–negative integers. Now, the area of each triangle is $1/2$, and so the area of the whole rectangle is rational. That is,

$$A = \left(a + b\sqrt{2}\right)\left(c + d\sqrt{2}\right)$$

is a rational number. If we expand the right–hand side, then all the $\sqrt{2}$ terms must cancel, and so $ad = -bc$. But ad and bc cannot be negative, so the only way for this last equation to be true is that $ad = bc = 0$. Therefore, we have

$$\begin{aligned}\frac{a + b\sqrt{2}}{c + d\sqrt{2}} &= \frac{\left(a + b\sqrt{2}\right)\left(c - d\sqrt{2}\right)}{c^2 - 2d^2} \\ &= \frac{(ac - 2bd) + (bc - ad)\sqrt{2}}{c^2 - 2d^2} = \frac{ac - 2bd}{c^2 - 2d^2}\,,\end{aligned}$$

which is rational, and we are finished.

Q1800 Let x, y, z be positive real numbers such that $xyz = 1$. Show that

$$\frac{x^2}{3 + (x-1)^2} + \frac{y^2}{3 + (y-1)^2} + \frac{z^2}{3 + (z-1)^2} \geq 1\,.$$

SOLUTION We shall use two very important inequalities among real numbers. The Cauchy–Schwarz Inequality states that

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2$$

for any real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$. The Arithmetic Mean–Geometric Mean Inequality is

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1a_2 \cdots a_n)^{1/n}$$

for any positive reals $a_1, a_2, \ldots, a_n$.

For brevity, we write the three denominators in the question as a, b, c. Use the Cauchy–Schwarz Inequality with $n = 3$, and choose the particular values

$$a_1 = \frac{x}{\sqrt{a}}\ ,\ a_2 = \frac{y}{\sqrt{b}}\ ,\ a_3 = \frac{z}{\sqrt{c}}\ , \quad b_1 = \sqrt{a}\ ,\ b_2 = \sqrt{b}\ ,\ b_3 = \sqrt{c}\ .$$

Then we have

$$\Big(\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c}\Big)(a+b+c) \geq (x+y+z)^2\ .$$

Therefore, we need to prove that $(x+y+z)^2 \geq a+b+c$; that is,

$$(x+y+z)^2 \geq 9 + (x-1)^2 + (y-1)^2 + (z-1)^2\ ;$$

some easy algebra shows that this is equivalent to

$$xy + yz + zx + x + y + z \geq 6\ . \tag{$*$}$$

Now we use the Arithmetic Mean-Geometric Mean Inequality for $n = 6$, taking

$$a_1 = xy\ ,\ a_2 = yz\ ,\ a_3 = zx\ ,\ a_4 = x\ ,\ a_5 = y\ ,\ a_6 = z\ .$$

We obtain

$$xy + yz + zx + x + y + z \geq 6\big((xy)(yz)(zx)xyz\big)^{1/6} = 6(xyz)^{1/2}\ ;$$

since $xyz = 1$, this proves $(*)$ and we are finished.